

Noetherian schemes over abelian symmetric monoidal categories

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Abstract

In this paper, we develop basic results of algebraic geometry over abelian symmetric monoidal categories. Let A be a commutative monoid object in an abelian symmetric monoidal category $(\mathbf{C}, \otimes, 1)$ satisfying certain conditions and let $\mathcal{E}(A) = \text{Hom}_{A\text{-Mod}}(A, A)$. If the subobjects of A satisfy a certain compactness property, we say that A is Noetherian. We study the localisation of A with respect to any $s \in \mathcal{E}(A)$ and define the quotient A/\mathcal{I} of A with respect to any ideal $\mathcal{I} \subseteq \mathcal{E}(A)$. We use this to develop appropriate analogues of the basic notions from usual algebraic geometry (such as Noetherian schemes, irreducible, integral and reduced schemes, function field, the local ring at the generic point of a closed subscheme, etc) for schemes over $(\mathbf{C}, \otimes, 1)$. Our notion of a scheme over a symmetric monoidal category $(\mathbf{C}, \otimes, 1)$ is that of Toën and Vaquié.

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1 Introduction

The relative algebraic geometry over a symmetric monoidal category $(\mathbf{C}, \otimes, 1)$ has been studied at several places in the literature (see, for instance, Deligne [8], Hakim [10], Toën and Vaquié [17]). When $\mathbf{C} = R\text{-Mod}$, the category of modules over an ordinary commutative ring R , this reduces to the usual algebraic geometry of schemes over $\text{Spec}(R)$. In this paper, we will develop basic results of commutative algebra and algebraic geometry over abelian symmetric monoidal categories satisfying certain conditions. For instance, our methods enable us to do algebraic geometry in the category of presheaves of abelian groups over a topological space. This paper continues our research program to study monoid objects and schemes over symmetric monoidal categories (see [2], [3], [4], [5], [6], [7]).

More precisely, let $(\mathbf{C}, \otimes, 1)$ be an abelian symmetric monoidal category satisfying certain conditions described in Section 2. We will use the notion of schemes over $(\mathbf{C}, \otimes, 1)$ introduced by Toën and Vaquié [17]. It is natural to ask if we can develop in detail the results of intersection theory for schemes over $(\mathbf{C}, \otimes, 1)$. A starting point for this is to define appropriate analogues of the

basic notions from usual algebraic geometry (such as Noetherian schemes, irreducible, integral and reduced schemes, function field, the local ring at the generic point of a closed subscheme, etc) for schemes over a symmetric monoidal category $(\mathbf{C}, \otimes, 1)$. This is the aim of the present paper. For our purposes, we will also need to develop some commutative algebra over $(\mathbf{C}, \otimes, 1)$ and this utilises the notion of localisation of commutative monoid objects introduced in [2]. For the sake of convenience, the main properties of this localisation developed in [2] are recalled briefly in Section 2.

Let $Comm(\mathbf{C})$ denote the category of commutative monoid objects in $(\mathbf{C}, \otimes, 1)$. For any commutative monoid object A , we let $A - Mod$ denote the category of A -modules. Following [17], we set $Aff_{\mathbf{C}} := Comm(\mathbf{C})^{op}$ to be the category of affine schemes over \mathbf{C} . The affine scheme corresponding to a commutative monoid object A will be denoted by $Spec(A)$. Given an element $s \in \mathcal{E}(A) := Hom_{A-Mod}(A, A)$, we consider the localisation A_s of A introduced in [2]. Then, in Section 2, we show that any morphism $Spec(A_s) \rightarrow Spec(A)$ is a Zariski open immersion. Further, we prove that a collection $\{Spec(A_{t_i}) \rightarrow Spec(A)\}_{t_i \in \mathcal{E}(A), i \in I}$ of Zariski open immersions forms a cover of $Spec(A)$ if and only if $\{t_i\}_{i \in I}$ generate the unit ideal in the ring $\mathcal{E}(A)$.

We start working with Noetherian schemes (see Definition 3.6) in Section 3. A commutative monoid object A is said to be Noetherian if its subobjects (in $A - Mod$) satisfy a certain compactness property (see Definition 3.1). As in usual algebraic geometry, we prove that being Noetherian is a local property of schemes. Given a Noetherian monoid A and any ideal $\mathcal{I} \subseteq \mathcal{E}(A)$, we introduce a “quotient monoid” A/\mathcal{I} which is used for the construction of closed subschemes in Section 5. If A is Noetherian, so is the quotient monoid A/\mathcal{I} and the canonical morphism $p : A \rightarrow A/\mathcal{I}$ is an epimorphism in the category $Comm(\mathbf{C})$. Moreover, we show that if A is a Noetherian monoid object, $\mathcal{E}(A)$ is an ordinary Noetherian commutative ring and $\mathcal{E}(A/\mathcal{I}) = \mathcal{E}(A)/\mathcal{I}$.

We consider integral schemes in Section 4. We show that an integral scheme X over $(\mathbf{C}, \otimes, 1)$ is reduced and irreducible. For our purposes, we will need to consider a second, related notion of integrality that we shall refer to as “weak integrality”. We show that a reduced and irreducible scheme is weakly integral. We then associate to any integral scheme X , a field $k(X)$ that plays the role of function field in the context of schemes over $(\mathbf{C}, \otimes, 1)$. Thereafter, given a dominant morphism $f : Y \rightarrow X$ of integral schemes over \mathbf{C} , we construct an induced morphism $k(f) : k(X) \rightarrow k(Y)$ of function fields.

Finally, in Section 5, we construct closed subschemes of a Noetherian and semi-separated scheme. More generally, we show that there is a one-one correspondence between quasi-coherent sheaves of algebras on a semi-separated scheme X and affine morphisms $Y \rightarrow X$ (see Definition 5.3 and Proposition 5.4). In particular, when we have a quasi-coherent sheaf of quotient monoids on a Noetherian and semi-separated scheme X , the corresponding affine morphism $Y \rightarrow X$ gives us a closed subscheme Y of X . Further, we show that the closed subscheme Y of X is also Noetherian. Finally, to any integral closed subscheme Y of a Noetherian, integral and semi-separated scheme X , we associate a local ring \mathcal{O}_Y . In usual algebraic geometry, \mathcal{O}_Y is the local ring at the generic point of the integral closed subscheme Y .

Section 6 is devoted to examples. If X is a topological space and \mathcal{A} is a presheaf of commutative rings on X , we show that our theory can be used to do algebraic geometry in the category of presheaves of \mathcal{A} -modules on X . We then use this fact to give several natural examples of our theory.

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2 Coverings of affine schemes

Let $(\mathbf{C}, \otimes, 1)$ be an abelian symmetric monoidal category. We assume that \mathbf{C} contains small limits and small colimits and for any object $X \in \mathbf{C}$, the functor $_ \otimes X$ preserves colimits. We let $Comm(\mathbf{C})$ denote the category of unital commutative monoids in \mathbf{C} . For an object A in $Comm(\mathbf{C})$, we will always denote by $m_A : A \otimes A \rightarrow A$ the “multiplication map” and by $e_A : 1 \rightarrow A$ the “unit map” on A . Further, for any object A in $Comm(\mathbf{C})$, we will denote by $A - Mod$ the category of modules over A . For generalities on monoids and modules over them in symmetric monoidal categories, we refer the reader to [11]. All monoid objects considered in this paper shall be assumed to be unital and commutative. Further, for any monoid object A , we will assume that filtered colimits commute with finite limits in $A - Mod$. This latter assumption is key to the results on localisations in [2], which we shall use throughout this paper.

Since \mathbf{C} is an abelian category, finite products and finite coproducts in \mathbf{C} coincide. For any object X in \mathbf{C} and any integer $r > 0$ we let X^r denote the finite product (or coproduct) of r -copies of X .

We will also assume that the category \mathbf{C} satisfies the following two technical conditions:

(C1) The unit object 1 is compact, i.e., the functor $Hom(1, _)$ on \mathbf{C} preserves filtered colimits. Since $Hom(1, M) \cong Hom_{A-Mod}(A, M)$ for any monoid A and any object M in $A - Mod$, it follows that A is a compact object of $A - Mod$. Further, we assume that given a finite system (not necessarily filtered) of objects of the form $\{A^{r_i}\}_{i \in I}$, $r_i \geq 0$, we have

$$colim_{i \in I} Hom_{A-Mod}(A, A^{r_i}) \cong Hom_{A-Mod}(A, colim_{i \in I} A^{r_i}) \quad (2.1)$$

Again, we note that (2.1) is equivalent to assuming that $colim_{i \in I} Hom(1, A^{r_i}) \cong Hom(1, colim_{i \in I} A^{r_i})$.

(C2) If $A \in Comm(\mathbf{C})$ is a commutative monoid, an object $M \in A - Mod$ will be said to be finitely presented if the functor $Hom_{A-Mod}(M, _)$ on $A - Mod$ preserves directed colimits. We will assume that for any commutative monoid A in \mathbf{C} , every object in $A - Mod$ may be expressed as a directed colimit of finitely presented objects in $A - Mod$.

Remark 2.1. The condition (C2) above may be seen as an analogue of the fact that in the category of modules over an ordinary commutative ring, any module may be expressed as a directed colimit of finitely generated submodules. In fact, the condition (C2) implies that the category $A - Mod$ is “locally finitely presented”. The theory of locally finitely presentable categories is fairly well developed in the literature and one may see, for instance, [9], [13], [14], [15], [16]. In [5], we have also studied module categories for monoid objects in symmetric monoidal categories under the somewhat similar assumption of “locally finitely generated”. For generalities on locally finitely generated and locally finitely presentable categories, we refer the reader to [1].

To any unital, commutative monoid object A in $(\mathbf{C}, \otimes, 1)$, we can associate the object $\mathcal{E}(A) := Hom_{A-Mod}(A, A)$ of A -module morphisms from A to A . It is well known, see, for instance [12],

that $\mathcal{E}(A)$ is a commutative ring. Given a morphism $g : A \rightarrow B$ in $Comm(\mathbf{C})$, it follows from base change that we have an induced morphism $\mathcal{E}(g) : \mathcal{E}(A) \rightarrow \mathcal{E}(B)$ of commutative rings.

Let A be a monoid object in $(\mathbf{C}, \otimes, 1)$ and let us choose any $t \in \mathcal{E}(A)$. Then, in [2, §3], we have defined a commutative monoid object A_t as follows:

$$A_t := \operatorname{colim}(A \xrightarrow{t} A \xrightarrow{t} A \xrightarrow{t} \dots) \quad (2.2)$$

which we call the localisation of A with respect to t . More generally, if $S \subseteq \mathcal{E}(A)$ is a “multiplicatively closed subset”, i.e., the identity map $1_A \in S$ and for any $s, t \in S$, the composition $s \circ t = t \circ s \in S$, we have defined the localisation

$$A_S := \operatorname{colim}_{s \in S} A_s \quad (2.3)$$

in [2]. We note here that since S is closed under composition, the colimit in (2.3) is filtered. The object A_S is equipped with a canonical morphism $I_S : A \rightarrow A_S$ of monoids. For any A -module M , the localisation of M with respect to S is defined to be $M_S := M \otimes_A A_S$. Further, we have shown in [2] that the localisation A_S satisfies the following properties:

(a) A_S is a flat A -module, i.e., the functor $-\otimes_A A_S$ on $A\text{-Mod}$ preserves finite limits and finite colimits.

(b) Consider the morphism $\mathcal{E}(I_S) : \mathcal{E}(A) \rightarrow \mathcal{E}(A_S)$ induced by the morphism $I_S : A \rightarrow A_S$. Then, for any $s \in S$, the morphism $\mathcal{E}(I_S)(s) \in \mathcal{E}(A_S)$ is an isomorphism. Further, given any morphism $g : A \rightarrow B$ in $Comm(\mathbf{C})$ such that $\mathcal{E}(g)(s) \in \mathcal{E}(B)$ is an isomorphism for each $s \in S$, there exists a unique morphism $h : A_S \rightarrow B$ such that $g = h \circ I_S$.

We note that property (b) above implies that the canonical morphism $I_S : A \rightarrow A_S$ is an epimorphism in the category $Comm(\mathbf{C})$, i.e., given any morphisms $f_1, f_2 : A_S \rightarrow B$ in $Comm(\mathbf{C})$ such that $f_1 \circ I_S = f_2 \circ I_S$, we must have $f_1 = f_2$.

Let $Aff_{\mathbf{C}} = Comm(\mathbf{C})^{op}$ denote the category of affine schemes over \mathbf{C} . If A is an object of $Comm(\mathbf{C})$, we will often use $Spec(A)$ to denote the corresponding object in $Aff_{\mathbf{C}}$. Then, Toën and Vaquié (see [17, Définition 2.10]) have introduced the notion of Zariski coverings in the category $Aff_{\mathbf{C}}$, determining a Grothendieck site that is also subcanonical, i.e. the representable presheaves on $Aff_{\mathbf{C}}$ are also sheaves. Accordingly, let $Sh(Aff_{\mathbf{C}})$ denote the category of sheaves of sets on $Aff_{\mathbf{C}}$. Further, in [17, Définition 2.12], Toën and Vaquié have introduced a suitable notion of Zariski open immersions in the category $Sh(Aff_{\mathbf{C}})$ that is stable under composition and base change. Then, a scheme X over \mathbf{C} is defined to be an object of $Sh(Aff_{\mathbf{C}})$ admitting a Zariski covering by affine schemes (see [17, Définition 2.15]). By abuse of notation, we will often denote the sheaf on $Aff_{\mathbf{C}}$ represented by a scheme X (resp. an affine scheme $Spec(A)$) also by X (resp. $Spec(A)$).

Given a monoid A , in this section, our aim is to study Zariski coverings of $Spec(A)$ by means of schemes of the form $\{Spec(A_t)\}_{t \in \mathcal{E}(A)}$. We recall here the notion of a Zariski open immersion of affine schemes over $(\mathbf{C}, \otimes, 1)$ as defined in [17, Définition 2.9].

Definition 2.2. Let $f : A \rightarrow B$ be a morphism in $Comm(\mathbf{C})$.

(1) The morphism f is flat if the functor $-\otimes_A B : A\text{-Mod} \rightarrow B\text{-Mod}$ is exact, i.e., preserves finite limits.

(2) The morphism f is an epimorphism if, for all A' in $\text{Comm}(\mathbf{C})$, the induced morphism $f^* : \text{Hom}_{\text{Comm}(\mathbf{C})}(B, A') \longrightarrow \text{Hom}_{\text{Comm}(\mathbf{C})}(A, A')$ is injective.

(3) The morphism f is of finite presentation if for any filtered system of objects $A'_i \in A/\text{Comm}(\mathbf{C})$, $i \in I$ the natural isomorphism

$$\text{colim}_{i \in I} \text{Hom}_{A/\text{Comm}(\mathbf{C})}(B, A'_i) \longrightarrow \text{Hom}_{A/\text{Comm}(\mathbf{C})}(B, \text{colim}_{i \in I} A'_i) \quad (2.4)$$

is an isomorphism.

(4) The morphism $\text{Spec}(B) \longrightarrow \text{Spec}(A)$ induced by f is a Zariski open immersion if f is a flat epimorphism of finite presentation.

(5) Given morphisms $f_i : A \longrightarrow B_i$, $i \in I$, the collection of functors $\{_ \otimes_A B_i : A\text{-Mod} \longrightarrow B_i\text{-Mod}\}_{i \in I}$ is said to be conservative if, for any M in $A\text{-Mod}$, $M = 0$ if and only if $M \otimes_A B_i = 0$ for each $i \in I$.

(6) Consider morphisms $f_i : A \longrightarrow B_i$, $i \in I$ such that there exists a finite subcollection $I' \subseteq I$ such that the family of functors $\{_ \otimes_A B_i : A\text{-Mod} \longrightarrow B_i\text{-Mod}\}_{i \in I'}$ is conservative. Then, if each $f_i : A \longrightarrow B_i$, $i \in I$ induces a Zariski open immersion of affine schemes, $\{\text{Spec}(B_i) \longrightarrow \text{Spec}(A)\}_{i \in I}$ is said to be a Zariski open cover of $\text{Spec}(A)$.

We also recall here the definition of a scheme over $(\mathbf{C}, \otimes, 1)$ due to Toën and Vaquié (see [17, Définition 2.15]).

Definition 2.3. Let X be an object of $\text{Sh}(\text{Aff}_{\mathbf{C}})$. Then, X is a scheme over $(\mathbf{C}, \otimes, 1)$ if there exists a family $\{X_i\}_{i \in I}$ of affine schemes over $(\mathbf{C}, \otimes, 1)$ and a morphism

$$p : \coprod_{i \in I} X_i \longrightarrow X \quad (2.5)$$

satisfying the following conditions:

(a) The morphism p is an epimorphism in $\text{Sh}(\text{Aff}_{\mathbf{C}})$.

(b) For each $i \in I$, the morphism $X_i \longrightarrow X$ is a Zariski open immersion in $\text{Sh}(\text{Aff}_{\mathbf{C}})$.

Lemma 2.4. Let A_i , $i \in I$ be a filtered system of objects in $\text{Comm}(\mathbf{C})$ and let $A = \text{colim}_{i \in I} A_i$. Then, we have $\mathcal{E}(A) = \text{colim}_{i \in I} \mathcal{E}(A_i)$.

Proof. By assumption (C1), 1 is a compact object of \mathbf{C} and it follows therefore that:

$$\mathcal{E}(A) = \text{Hom}_{A\text{-Mod}}(A, A) \cong \text{Hom}(1, A) \cong \text{colim}_{i \in I} \text{Hom}(1, A_i) \cong \text{Hom}_{A_i\text{-Mod}}(A_i, A_i) = \mathcal{E}(A_i)$$

This proves the result. \square

Proposition 2.5. Let A be an object of $\text{Comm}(\mathbf{C})$. We choose some $t \in \mathcal{E}(A)$ and let $I_t : A \longrightarrow A_t$ denote the localisation of A with respect to t . Then, the induced morphism $\text{Spec}(A_t) \longrightarrow \text{Spec}(A)$ is a Zariski open immersion of schemes.

Proof. We have to verify conditions (1), (2) and (3) in Definition 2.2 for the morphism f . We have already mentioned that the functor $_ \otimes_A A_t$ preserves finite limits. Similarly, we have also mentioned before that any morphism $I_t : A \longrightarrow A_t$ induced by a localisation is an epimorphism in $Comm(\mathbf{C})$. It follows that f satisfies conditions (1) and (2) of Definition 2.2.

Finally, we consider a filtered system $A'_i \in A/Comm(\mathbf{C})$, $i \in I$ and we set $A' := \text{colim}_{i \in I} A'_i$. By definition, each object $A'_i \in A/Comm(\mathbf{C})$ is equipped with canonical morphisms

$$h_i : A \longrightarrow A'_i \quad g_i : A'_i \longrightarrow A' \quad (2.6)$$

such that $g_i \circ h_i = g_j \circ h_j \ \forall i, j \in I$. We consider a morphism $g : A_t \longrightarrow A'$ in $A/Comm(\mathbf{C})$. Then,

$$g \circ I_t = g_i \circ h_i : A \longrightarrow A' \quad \forall i \in I \quad (2.7)$$

Hence, for all $i \in I$,

$$\mathcal{E}(g_i \circ h_i) = \mathcal{E}(g \circ I_t)(t) = \mathcal{E}(g)(\mathcal{E}(I_t)(t)) \quad (2.8)$$

is a unit in $\mathcal{E}(A')$. From Lemma 2.4, we have $\mathcal{E}(A') = \text{colim}_{i \in I} \mathcal{E}(A'_i)$ and hence there exists $i_0 \in I$ such that $\mathcal{E}(h_{i_0})(t)$ is a unit in $\mathcal{E}(A'_{i_0})$. It now follows that there exists a morphism $h' : A_t \longrightarrow A'_{i_0}$ such that $h_{i_0} = h' \circ I_t$. It follows that f satisfies condition (3) of Definition 2.2. \square

Lemma 2.6. (a) Let A be a commutative monoid and let M be an A -module. Let $\{t_i\}_{i \in I}$ be a finite collection of elements $t_i \in \mathcal{E}(A)$ such that $\sum_{i \in I} t_i = 1$ and let M_i denote the respective localisations $M_i := M_{t_i}$, $\forall i \in I$. Then, if each $M_i = 0$, we must have $M = 0$.

(b) Let A be a commutative monoid and let M be an A -module. Let $\{t_i\}_{i \in I}$ be a finite collection of elements $t_i \in \mathcal{E}(A)$ such that there exists a collection $\{s_i\}_{i \in I}$, $s_i \in \mathcal{E}(A)$ such that $\sum_{i \in I} s_i t_i = 1$. Let M_i denote the respective localisations $M_i := M_{t_i}$, $\forall i \in I$. Then, if each $M_i = 0$, we must have $M = 0$.

Proof. (a) For any $t \in \mathcal{E}(A)$, we denote the induced morphism $M \cong M \otimes_A A \xrightarrow{1 \otimes t} M \otimes_A A \cong M$ by t_M . By definition, we know that

$$M_t := M \otimes_A A_t = \text{colim}(M \xrightarrow{t_M} M \xrightarrow{t_M} M \xrightarrow{t_M} \dots) \quad (2.9)$$

(since $M \otimes_A _$ commutes with colimits). Let N be a finitely presented object in $A - Mod$. We note that $Hom_{A-Mod}(N, M)$ can be made into an $\mathcal{E}(A)$ -module as follows: given $f \in Hom_{A-Mod}(N, M)$, $t \in \mathcal{E}(A)$, we set $f \cdot t := t_M \circ f \in Hom_{A-Mod}(N, M)$.

In particular, it is clear that the induced morphism $Hom_{A-Mod}(N, t_M) : Hom_{A-Mod}(N, M) \longrightarrow Hom_{A-Mod}(N, M)$ is identical to multiplication by $t \in \mathcal{E}(A)$ on the $\mathcal{E}(A)$ -module $Hom_{A-Mod}(N, M)$. Since N is a finitely presented object of $A - Mod$, it follows that

$$\begin{aligned} & Hom_{A-Mod}(N, M_t) \\ & \cong \text{colim}_{i \in I} \left(Hom_{A-Mod}(N, M) \xrightarrow{Hom_{A-Mod}(N, t_M)} Hom_{A-Mod}(N, M) \xrightarrow{Hom_{A-Mod}(N, t_M)} \dots \right) \\ & \cong \text{colim}_{i \in I} \left(Hom_{A-Mod}(N, M) \xrightarrow{\cdot t} Hom_{A-Mod}(N, M) \xrightarrow{\cdot t} \dots \right) \\ & \cong Hom_{A-Mod}(N, M)_t \end{aligned} \quad (2.10)$$

where $Hom_{A-Mod}(N, M)_t$ denotes the localisation of the $\mathcal{E}(A)$ -module $Hom_{A-Mod}(N, M)$ with respect to $t \in \mathcal{E}(A)$.

Hence, if $M_i = M_{t_i} = 0$ for all $i \in I$, it follows from (2.10) that for any finitely presented object N in $A-Mod$, $Hom_{A-Mod}(N, M)_{t_i} = 0$. Since $\sum_{i \in I} t_i = 1$ in $\mathcal{E}(A)$, it follows that $Hom_{A-Mod}(N, M) = 0$ for any finitely presented object N in $A-Mod$. Finally, since any object in $A-Mod$ can be expressed as a colimit of finitely presented objects (using condition (C2)), it follows that $Hom_{A-Mod}(N, M) = 0$ for any object N in $A-Mod$. Hence, $M = 0$.

(b) By interchanging colimits, it follows from (2.9) that for any $i \in I$, $M_{s_i t_i} = (M_{t_i})_{s_i}$. Then, since each $M_{t_i} = 0$, we have $M_{s_i t_i} = 0$ for each $i \in I$. It now follows from part (a) that $M = 0$. \square

Henceforth, given a monoid A , we will say that a finite collection $\{t_i\}_{i \in I}$ of elements $t_i \in \mathcal{E}(A)$ is a partition of unity on A if there exist elements $\{s_i\}_{i \in I}$, $s_i \in \mathcal{E}(A)$ such that $\sum_{i \in I} s_i t_i = 1$.

Proposition 2.7. *Let A be a commutative monoid and let $u : M \rightarrow N$ be a morphism of A -modules. Let $\{t_i\}_{i \in I}$ be a partition of unity on A . For any $i \in I$, let us denote by $u_i : M_i := M_{t_i} \rightarrow N_i := N_{t_i}$ the induced morphisms on the localisations of M and N with respect to t_i . Then, $u : M \rightarrow N$ is an isomorphism if and only if each $u_i : M_i \rightarrow N_i$, $i \in I$ is an isomorphism.*

Proof. The “only if” part of the result is clear. Conversely, suppose that each $u_i : M_i \rightarrow N_i$ is an isomorphism. We consider the objects $Ker(u)$ and $Coker(u)$ in $A-Mod$ defined as follows:

$$Ker(u) := \lim(M \xrightarrow{u} N \leftarrow 0) \quad Coker(u) := \text{colim}(N \xleftarrow{u} M \rightarrow 0) \quad (2.11)$$

Since each $A_i := A_{t_i}$ is a flat A -module, i.e., the functor $- \otimes_A A_i$ preserves finite limits and finite colimits, it follows that for any $i \in I$, we have:

$$Ker(u_i) = \lim(M_i \xrightarrow{u_i} N_i \leftarrow 0) = \lim(M \otimes_A A_i \xrightarrow{u_i} N \otimes_A A_i \leftarrow 0) = Ker(u)_{t_i} \quad (2.12)$$

Similarly, for any $i \in I$, we have

$$Coker(u_i) = \lim(M_i \xrightarrow{u_i} N_i \leftarrow 0) = \lim(M \otimes_A A_i \xrightarrow{u_i} N \otimes_A A_i \leftarrow 0) = Coker(u)_{t_i} \quad (2.13)$$

Since each u_i is an isomorphism, we have $Ker(u_i) = Coker(u_i) = 0$ for each $i \in I$. It follows from (2.12) that $Ker(u)_{t_i} = Coker(u)_{t_i} = 0$ for each $i \in I$. Combining with Lemma 2.6, it follows that $Ker(u) = Coker(u) = 0$. Since $A-Mod$ is an abelian category, $u : M \rightarrow N$ is an isomorphism. \square

Corollary 2.8. *Let A be a commutative monoid object in \mathbf{C} and let $S \subseteq \mathcal{E}(A)$ be a multiplicatively closed set. Then, $\mathcal{E}(A_S) = \mathcal{E}(A)_S$.*

Proof. From condition (C1), we know that A is a compact object of $A-Mod$. Then, it follows from (2.10) that $Hom_{A-Mod}(A, A_s) \cong Hom_{A-Mod}(A, A)_s \cong \mathcal{E}(A)_s$ for any $s \in \mathcal{E}(A)$. Hence, we have:

$$\begin{aligned} \mathcal{E}(A_S) &\cong Hom_{A_S-Mod}(A_S, A_S) \cong Hom_{A-Mod}(A, A_S) \cong Hom_{A-Mod}(A, \text{colim}_{s \in S} A_s) \\ &\cong \text{colim}_{s \in S} Hom_{A-Mod}(A, A_s) \cong \text{colim}_{s \in S} \mathcal{E}(A)_s \cong \mathcal{E}(A)_S \end{aligned} \quad (2.14)$$

where the isomorphism $Hom_{A-Mod}(A, \text{colim}_{s \in S} A_s) \cong \text{colim}_{s \in S} Hom_{A-Mod}(A, A_s)$ in (2.14) follows from the fact that A is compact in $A-Mod$. \square

Proposition 2.9. *Let A be a commutative monoid object in \mathbf{C} and let $\{t_i\}_{i \in I}$ be a partition of unity on A . Then, the schemes $\{Spec(A_{t_i})\}_{i \in I}$ form a Zariski open cover of $Spec(A)$.*

Proof. From Proposition 2.5, we know that each morphism $Spec(A_{t_i}) \rightarrow Spec(A)$ is a Zariski open immersion. Further, from Proposition 2.7, we know that the collection of functors $\{- \otimes_A A_{t_i} : A - Mod \rightarrow A_{t_i} - Mod\}_{i \in I}$ is conservative. It follows that the collection $\{Spec(A_{t_i}) \rightarrow Spec(A)\}_{i \in I}$ is a Zariski open cover of $Spec(A)$. \square

We conclude this section by proving the converse of Proposition 2.9. Given a monoid A and some $t \in \mathcal{E}(A)$, we define:

$$A/tA := colim(A \xleftarrow{t} A \rightarrow 0) \quad (2.15)$$

It is easy to check that A/tA is also a commutative monoid. Further, using assumption (C1), we have

$$Hom_{A-Mod}(A, A/tA) \cong colim(Hom_{A-Mod}(A, A) \xleftarrow{t} Hom_{A-Mod}(A, A) \rightarrow 0) \cong \mathcal{E}(A)/(t) \quad (2.16)$$

where (t) in (2.16) denotes the principal ideal in $\mathcal{E}(A)$ generated by $t \in \mathcal{E}(A)$. It follows that:

$$\mathcal{E}(A/tA) = Hom_{A/tA-Mod}(A/tA, A/tA) \cong Hom_{A-Mod}(A, A/tA) \cong \mathcal{E}(A)/(t) \quad (2.17)$$

More generally, if $\{t_1, \dots, t_n\}$ is a set of elements in $\mathcal{E}(A)$, we know from (2.15) and (2.17) that the monoid A/t_1A has $\mathcal{E}(A/t_1A) \cong \mathcal{E}(A)/(t_1)$. Then, $t_2 \in \mathcal{E}(A)$ defines a class \bar{t}_2 in $\mathcal{E}(A)/(t_1) \cong \mathcal{E}(A/t_1A)$ and we set

$$A/(t_1, t_2)A := colim(A/t_1A \xleftarrow{\bar{t}_2} A/t_1A \rightarrow 0) \quad (2.18)$$

Again, $A/(t_1, t_2)A$ is a monoid and using (2.17), we conclude that

$$\mathcal{E}(A/(t_1, t_2)A) \cong \mathcal{E}(A/t_1A)/(\bar{t}_2) \cong \mathcal{E}(A)/(t_1, t_2) \quad (2.19)$$

where (t_1, t_2) is the ideal in $\mathcal{E}(A)$ generated by t_1 and t_2 . More generally, for each $2 \leq i \leq n$, we set

$$A/(t_1, \dots, t_i)A := colim(A/(t_1, \dots, t_{i-1})A \xleftarrow{\bar{t}_i} A/(t_1, \dots, t_{i-1})A \rightarrow 0) \quad (2.20)$$

and note that $\mathcal{E}(A/(t_1, \dots, t_i)A) \cong \mathcal{E}(A)/(t_1, \dots, t_i)$.

Proposition 2.10. *Let A be an object of $Comm(\mathbf{C})$ and let $\{t_i\}_{i \in I}$ be a collection of elements $t_i \in \mathcal{E}(A)$ such that the collection $\{Spec(A_{t_i}) \rightarrow Spec(A)\}_{i \in I}$ forms a Zariski open cover of $Spec(A)$. Then, there exists a finite subcollection $\{t_i\}_{i \in I'}$, $I' \subseteq I$ that is a partition of unity on A .*

Proof. By definition, since $\{Spec(A_{t_i}) \rightarrow Spec(A)\}_{i \in I}$ forms a Zariski open cover of $Spec(A)$, there is a finite subcollection $I' = \{1, 2, \dots, n\} \subseteq I$ such that the collection of functors $\{- \otimes_A A_{t_i} : A - Mod \rightarrow A_{t_i} - Mod\}_{i \in I'}$ is conservative. We will show that $\{t_1, t_2, \dots, t_n\}$ is a partition of unity on A . For any chosen $j \in \{1, 2, \dots, n\}$, we have:

$$A/(t_1, \dots, t_j)A \otimes_A A_{t_j} \cong colim(A \xleftarrow{t_j} A \rightarrow 0) \otimes_A A_{t_j} \otimes_A A/(t_1, \dots, t_{j-1})A = 0 \quad (2.21)$$

From (2.20), it now follows that $A/(t_1, \dots, t_n)A \otimes_A A_{t_j} = 0$ and for each $1 \leq j \leq n$. Since the collection of functors $\{- \otimes_A A_{t_i} : A - Mod \rightarrow A_{t_i} - Mod\}_{i \in I'}$ is conservative, it follows that $A/(t_1, \dots, t_n)A = 0$. Hence, $\mathcal{E}(A/(t_1, \dots, t_n)A) = \mathcal{E}(A)/(t_1, \dots, t_n) = 0$. Hence, $\{t_1, \dots, t_n\}$ forms a partition of unity on A . \square

3 Noetherian monoids over $(\mathbf{C}, \otimes, 1)$

In this section, we will begin to describe the properties of Noetherian monoids and Noetherian schemes over $(\mathbf{C}, \otimes, 1)$ (see Definition 3.1 and Definition 3.6). As in usual algebraic geometry, we show that being Noetherian is a local property for schemes over $(\mathbf{C}, \otimes, 1)$. In Section 2, for any monoid A and any $t \in \mathcal{E}(A)$, we have already described the “quotient monoid” A/tA . We will extend this definition further to introduce, for any ideal $\mathcal{I} \subseteq \mathcal{E}(A)$, a “quotient monoid” A/\mathcal{I} . When A is Noetherian, we show that any quotient A/\mathcal{I} is also Noetherian. These results will put in place the basic framework for construction of closed subschemes of a Noetherian scheme X over $(\mathbf{C}, \otimes, 1)$, which will ultimately be done in Section 5. We start by defining Noetherian monoids in $(\mathbf{C}, \otimes, 1)$.

Definition 3.1. *Let A be a commutative monoid object in $(\mathbf{C}, \otimes, 1)$.*

(a) *Let $\{M_i\}_{i \in I}$ be a filtered inductive system of objects of $A - \text{Mod}$ connected by monomorphisms and let $M := \text{colim}_{i \in I} M_i$. Then, an object N of $A - \text{Mod}$ will be said to be finitely generated if the canonical map*

$$\text{colim}_{i \in I} \text{Hom}_{A - \text{Mod}}(N, M_i) \longrightarrow \text{Hom}_{A - \text{Mod}}(N, M) \quad (3.1)$$

is a bijection.

(b) *The monoid A will be said to be Noetherian if every subobject of A in $A - \text{Mod}$ is finitely generated in $A - \text{Mod}$.*

We remark here that our notion of Noetherian in Definition 3.1 is different from the notion of Noetherian considered in [5] which was further modified in [6].

Lemma 3.2. *Let $f : A \longrightarrow B$ be an epimorphism of monoids in $(\mathbf{C}, \otimes, 1)$. Then, we have an isomorphism $B \cong B \otimes_A B$.*

Proof. Suppose that we have morphisms $f_1, f_2 : B \longrightarrow C$ in $\text{Comm}(\mathbf{C})$ such that $f_1 \circ f = f_2 \circ f : A \longrightarrow C$. Since $f : A \longrightarrow B$ is an epimorphism in $\text{Comm}(\mathbf{C})$, it follows that $f_1 = f_2$. Hence, B is the pushout of the diagram $B \xleftarrow{f} A \xrightarrow{f} B$ in $\text{Comm}(\mathbf{C})$. Moreover, we know that for any given morphisms $g : D \longrightarrow D'$, $h : D \longrightarrow D''$ of objects in $\text{Comm}(\mathbf{C})$, the following is a pushout diagram in $\text{Comm}(\mathbf{C})$ (see, for instance, [3, Lemma 2.3])

$$\begin{array}{ccc} D & \xrightarrow{g} & D' \\ h \downarrow & & \downarrow \\ D'' & \longrightarrow & D' \otimes_D D'' \end{array} \quad (3.2)$$

In particular, it follows from (3.2) that $B \otimes_A B$ is the pushout of the diagram $B \xleftarrow{f} A \xrightarrow{f} B$ in $\text{Comm}(\mathbf{C})$. Hence, we have $B \cong B \otimes_A B$. \square

Proposition 3.3. *Let A be a commutative monoid object in $(\mathbf{C}, \otimes, 1)$ and let $f : A \longrightarrow B$ be a morphism of monoids inducing a Zariski open immersion of affine schemes. Then, if A is Noetherian, so is B .*

Proof. From Definition 2.2, we know that $f : A \longrightarrow B$ is an epimorphism of monoids. Hence, from Lemma 3.2 we know that $B \cong B \otimes_A B$. It follows that, for any B -module M , we have

$$M \otimes_A B \cong M \otimes_B B \otimes_A B \cong M \otimes_B B \cong M \quad (3.3)$$

Let J be a subobject of B in $B - Mod$. We consider the following pullback square in $A - Mod$:

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow \\ J & \longrightarrow & B \end{array} \quad (3.4)$$

Then, I is a subobject of A in $A - Mod$. Since A is Noetherian, I is a finitely generated object of $A - Mod$. Since B is a flat A -module, the following is also a pullback square:

$$\begin{array}{ccc} I \otimes_A B & \longrightarrow & A \otimes_A B \cong B \\ \downarrow & & \cong \downarrow \\ J \otimes_A B \cong J & \longrightarrow & B \otimes_A B \cong B \end{array} \quad (3.5)$$

where the isomorphism $J \otimes_A B \cong J$ appearing in (3.5) follows from (3.3). Since the right vertical arrow in the pullback square (3.5) is an isomorphism, we have $J \cong I \otimes_A B$. Now, suppose that $\{M_i\}_{i \in I}$ is a filtered inductive system of objects in $B - Mod$ connected by monomorphisms and let $M := \text{colim}_{i \in I} M_i$. Then, it follows that

$$\begin{aligned} \text{colim}_{i \in I} \text{Hom}_{B-Mod}(J, M_i) &\cong \text{colim}_{i \in I} \text{Hom}_{B-Mod}(I \otimes_A B, M_i) \\ &\cong \text{colim}_{i \in I} \text{Hom}_{A-Mod}(I, M_i) \cong \text{Hom}_{A-Mod}(I, M) \\ &\cong \text{Hom}_{B-Mod}(J, M) \end{aligned} \quad (3.6)$$

where the isomorphism $\text{colim}_{i \in I} \text{Hom}_A(I, M_i) \cong \text{Hom}_A(I, M)$ appearing in (3.6) follows from the fact that I is a finitely generated object of $A - Mod$. Hence, J is a finitely generated object of $B - Mod$. It follows that B is Noetherian. \square

Proposition 3.4. *Let A be a Noetherian commutative monoid object in $(\mathbf{C}, \otimes, 1)$. Then, $\mathcal{E}(A)$ is a Noetherian ring.*

Proof. Suppose that $\mathcal{E}(A)$ is non-Noetherian. Then, there exists a sequence $\{t_i\}_{i \in \mathbb{N}}$ of elements of $\mathcal{E}(A)$ such that we have a strictly increasing chain of ideals:

$$(t_1) \subsetneq (t_1, t_2) \subsetneq (t_1, t_2, t_3) \subsetneq \dots \quad (3.7)$$

in $\mathcal{E}(A)$ that does not stabilise. Let \mathcal{I}_i be the ideal generated by the elements $\{t_1, t_2, \dots, t_i\}$. For each $i \in \mathbb{N}$, we define:

$$I_i := \text{lim}(A \longrightarrow A/t_1 A \otimes_A A/t_2 A \otimes_A \dots \otimes_A A/t_i A \longleftarrow 0) \quad (3.8)$$

It is clear that we have a chain of subobjects of A in $A - Mod$ as follows

$$I_1 \xrightarrow{h_1} I_2 \xrightarrow{h_2} I_3 \xrightarrow{h_3} \dots \quad (3.9)$$

with each I_i a subobject of I_{i+1} . Suppose that for some given $i_0 \in \mathbb{N}$, we have

$$A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_{i_0} A \cong A/(t_1, \dots, t_{i_0}) A \quad (3.10)$$

Then, it follows that:

$$\begin{aligned} & A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_{i_0+1} A \\ & \cong A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_{i_0} A \otimes_A \operatorname{colim}(A \xleftarrow{t_{i_0+1}} A \longrightarrow 0) \\ & \cong \operatorname{colim}(A/t_1 A \otimes_A \cdots \otimes_A A/t_{i_0} A \xleftarrow{t_{i_0+1}} A/t_1 A \otimes_A \cdots \otimes_A A/t_{i_0} A \longrightarrow 0) \\ & \cong \operatorname{colim}(A/(t_1, \dots, t_{i_0}) A \xleftarrow{t_{i_0+1}} A/(t_1, \dots, t_{i_0}) A \longrightarrow 0) \\ & \cong A/(t_1, \dots, t_{i_0+1}) A \end{aligned} \quad (3.11)$$

Using induction, it follows from (3.11) that

$$A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_i A \cong A/(t_1, \dots, t_i) A \quad \forall i \in \mathbb{N} \quad (3.12)$$

From (3.12), it follows that

$$\begin{aligned} & \operatorname{Hom}_{A-\operatorname{Mod}}(A, A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_i A) \\ & \cong \operatorname{Hom}_{A-\operatorname{Mod}}(A, A/(t_1, t_2, \dots, t_i) A) \\ & \cong \operatorname{Hom}_{A/t_1 A-\operatorname{Mod}}(A/t_1 A, A/(t_1, t_2, \dots, t_i) A) \\ & \cong \dots \\ & \cong \operatorname{Hom}_{A/(t_1, t_2, \dots, t_{i-1}) A-\operatorname{Mod}}(A/(t_1, \dots, t_{i-1}) A, A/(t_1, t_2, \dots, t_i) A) \\ & \cong \operatorname{Hom}_{A/(t_1, t_2, \dots, t_i) A-\operatorname{Mod}}(A/(t_1, t_2, \dots, t_i) A, A/(t_1, t_2, \dots, t_i) A) \cong \mathcal{E}(A/(t_1, t_2, \dots, t_i) A) \end{aligned} \quad (3.13)$$

From the discussion preceding Proposition 2.10, we know that

$$\mathcal{E}(A/(t_1, t_2, \dots, t_i) A) \cong \mathcal{E}(A)/(t_1, \dots, t_i) \quad (3.14)$$

where (t_1, \dots, t_i) in (3.14) denotes the ideal in $\mathcal{E}(A)$ generated by $\{t_1, \dots, t_i\}$. From (3.13) and (3.14), it follows that for any $i \in \mathbb{N}$,

$$\begin{aligned} & \operatorname{Hom}_{A-\operatorname{Mod}}(A, I_i) \cong \operatorname{Hom}_{A-\operatorname{Mod}}(A, \lim(A \longrightarrow A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_i A \longleftarrow 0)) \\ & \cong \lim(\operatorname{Hom}_{A-\operatorname{Mod}}(A, A) \longrightarrow \operatorname{Hom}_{A-\operatorname{Mod}}(A, A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_i A) \longleftarrow 0) \\ & \cong \lim(\mathcal{E}(A) \longrightarrow \mathcal{E}(A)/(t_1, \dots, t_i) \longleftarrow 0) \cong (t_1, t_2, \dots, t_i) = \mathcal{J}_i \subseteq \mathcal{E}(A) \end{aligned} \quad (3.15)$$

We now consider the chain of subobjects $\{I_i\}_{i \in \mathbb{N}}$ described in (3.9) and set $I = \operatorname{colim}_{i \in \mathbb{N}} I_i$. For any i , let $h'_i : I_i \longrightarrow I$ be the canonical morphism from I_i to the colimit I . Since A is Noetherian, I is a finitely generated object of $A-\operatorname{Mod}$. Hence,

$$\operatorname{Hom}_{A-\operatorname{Mod}}(I, I) \cong \operatorname{colim}_{i \in \mathbb{N}} \operatorname{Hom}_{A-\operatorname{Mod}}(I, I_i) \quad (3.16)$$

In particular, it follows from (3.16) that there exists $j \in \mathbb{N}$ such that for each $i \geq j$, there is a morphism $g_i : I \longrightarrow I_i$ such that $1 = h'_i \circ g_i : I \xrightarrow{g_i} I_i \xrightarrow{h'_i} I$. Since each $h_i : I_i \longrightarrow I_{i+1}$ in (3.9) is a monomorphism, so is each morphism $h'_i : I_i \longrightarrow I$ to the colimit I . However, since $1 = g_i \circ h'_i$ for any $i \geq j$, it follows that h'_i is also an epimorphism and hence $h'_i : I_i \longrightarrow I$ is an isomorphism for $i \geq j$ ($A-\operatorname{Mod}$ being an abelian category). It follows that each $h_i : I_i \longrightarrow I_{i+1}$, $i \geq j$ is an isomorphism. Combining with (3.15), it follows therefore, that for each $i \geq j$, we have

$$\mathcal{J}_i = \operatorname{Hom}_A(A, I_i) = \operatorname{Hom}_A(A, I_{i+1}) = \mathcal{J}_{i+1} \quad (3.17)$$

which is a contradiction. Hence, $\mathcal{E}(A)$ is Noetherian. □

Proposition 3.5. *Let B be a commutative monoid object in $(\mathbf{C}, \otimes, 1)$ such that there exists a finite Zariski covering $\{\text{Spec}(A_i) \rightarrow \text{Spec}(B)\}_{i \in I}$ with each A_i , $i \in I$ a Noetherian monoid. Then, B is Noetherian.*

Proof. We consider a filtered inductive system of B -modules $\{M_k\}_{k \in K}$ connected by monomorphisms and set $M := \text{colim}_{k \in K} M_k$. Let $h_k : M_k \rightarrow M$ denote the canonical morphisms. Since $B\text{-Mod}$ is an abelian category, we know that a morphism $J \rightarrow B$ in $B\text{-Mod}$ defines a subobject of B if and only if:

$$0 = \lim(J \rightarrow B \leftarrow 0) \quad (3.18)$$

Let J be a subobject of B in $B\text{-Mod}$. Since each A_i , $i \in I$ is a flat B -module, we note that

$$0 = \lim(J \rightarrow B \leftarrow 0) \otimes_B A_i \cong \lim(J \otimes_B A_i \rightarrow A_i \leftarrow 0) \quad (3.19)$$

It follows from (3.19) that each $J \otimes_B A_i$, $i \in I$ is a subobject of A_i . Further, we note that for each $k \in K$, we have canonical morphisms

$$\text{Hom}_{B\text{-Mod}}(J, M_k) \rightarrow \text{Hom}_{B\text{-Mod}}(J, M) \quad (3.20)$$

induced by $h_k : M_k \rightarrow M$. Conversely, suppose that we have a morphism $f : J \rightarrow M$ in $B\text{-Mod}$. We consider the induced morphisms $f \otimes_B A_i : J \otimes_B A_i \rightarrow M \otimes_B A_i$ for each $i \in I$. Since each $h_k : M_k \rightarrow M$ is a monomorphism and A_i is a flat B -module, it follows that $\{M_k \otimes_B A_i\}_{k \in K}$ is also a filtered system connected by monomorphisms for each $i \in I$. Moreover, since each A_i is Noetherian, we know that

$$\text{Hom}_{A_i\text{-Mod}}(J \otimes_B A_i, M \otimes_B A_i) \cong \text{colim}_{k \in K} \text{Hom}_{A_i\text{-Mod}}(J \otimes_B A_i, M_k \otimes_B A_i) \quad (3.21)$$

Since I is finite and K is filtered, it follows that there exists some $k_0 \in K$ such that each morphism $f \otimes_B A_i : J \otimes_B A_i \rightarrow M \otimes_B A_i$, $i \in I$ factors through $M_{k_0} \otimes_B A_i$, i.e., there exist morphisms $g_{l,i} : J \otimes_B A_i \rightarrow M_l \otimes_B A_i \forall i \in I, l \geq k_0$ such that $(h_l \otimes_B A_i) \circ g_{l,i} = f \otimes_B A_i$. Then, for any $i, i' \in I$, it follows that

$$(h_l \otimes_B A_i \otimes_B A_{i'}) \circ (g_{l,i} \otimes_B A_{i'}) = f \otimes_B A_i \otimes_B A_{i'} = (h_l \otimes_B A_i \otimes_B A_{i'}) \circ (g_{l,i'} \otimes_B A_i) \quad (3.22)$$

as morphisms from $J \otimes_B A_i \otimes_B A_{i'}$ to $M \otimes_B A_i \otimes_B A_{i'}$. Since each $h_l : M_l \rightarrow M$ is a monomorphism and $A_i, A_{i'}$ are flat B -modules,

$$(h_l \otimes_B A_i \otimes_B A_{i'}) : M_l \otimes_B A_i \otimes_B A_{i'} \rightarrow M \otimes_B A_i \otimes_B A_{i'} \quad (3.23)$$

is a monomorphism. From (3.22) and (3.23), it follows that, for any $i, i' \in I$, we have

$$g_{l,i} \otimes_B A_{i'} = g_{l,i'} \otimes_B A_i : J \otimes_B A_i \otimes_B A_{i'} \rightarrow M_l \otimes_B A_i \otimes_B A_{i'} \quad (3.24)$$

Hence, for each $l \geq k_0$, using [17, Théorème 2.5], [17, Corollaire 2.11], we have an induced morphism

$$\begin{array}{ccc} J = \lim(\prod_{i \in I} J \otimes_B A_i \xrightarrow{\quad} \prod_{i, i' \in I} J \otimes_B A_i \otimes_B A_{i'}) & & \\ g_l \downarrow & & \\ M_l = \lim(\prod_{i \in I} M_l \otimes_B A_i \xrightarrow{\quad} \prod_{i, i' \in I} M_l \otimes_B A_i \otimes_B A_{i'}) & & \end{array} \quad (3.25)$$

where the limits in (3.25) are taken in $B\text{-Mod}$. It follows that the morphism $f : J \rightarrow M$ factors through M_l for each $l \geq k_0$. Hence, $\text{Hom}_{B\text{-Mod}}(J, M) \cong \text{colim}_{k \in K} \text{Hom}_{B\text{-Mod}}(J, M_k)$ and B is Noetherian. □

We are now ready to define a Noetherian scheme over $(\mathbf{C}, \otimes, 1)$.

Definition 3.6. Let X be a scheme over $(\mathbf{C}, \otimes, 1)$. Then, X is said to be Noetherian if, for any Zariski open immersion $U \rightarrow X$ with $U = \text{Spec}(A)$ affine, A is a Noetherian monoid.

Proposition 3.7. Let X be a scheme over $(\mathbf{C}, \otimes, 1)$ such that there exists a covering $\{U_i \rightarrow X\}_{i \in I}$ with $U_i = \text{Spec}(A_i)$ affine such that each A_i is Noetherian. Then, X is a Noetherian scheme.

Proof. We choose any Zariski open immersion $U \rightarrow X$ with $U = \text{Spec}(B)$ affine. Then, we consider the pullback squares

$$\begin{array}{ccc} X_i & \longrightarrow & U = \text{Spec}(B) \\ \downarrow & & \downarrow \\ U_i = \text{Spec}(A_i) & \longrightarrow & X \end{array} \quad (3.26)$$

We choose an affine covering $\{U_{ij} = \text{Spec}(A_{ij}) \rightarrow X_i\}_{i \in I, j \in J_i}$ for each $i \in I$. Since each $\text{Spec}(A_{ij})$, $j \in J_i$ admits a Zariski open immersion into $\text{Spec}(A_i)$, it follows from Proposition 3.3 that each A_{ij} is a Noetherian monoid. Further, it is clear that the collection $\{\text{Spec}(A_{ij}) \rightarrow \text{Spec}(B)\}_{j \in J_i, i \in I}$ (and hence a finite subcollection thereof) is a Zariski covering of $\text{Spec}(B)$. It now follows from Proposition 3.5 that B is a Noetherian monoid. This proves the result. \square

Given a monoid A and any $t \in \mathcal{E}(A)$, the construction of the “quotient monoid” A/tA has been introduced in (2.15). We will now generalise this construction. Let $\mathcal{I} \subseteq \mathcal{E}(A)$ be an ideal. For each $t \in \mathcal{I}$, we can consider the quotient A/tA as defined in (2.15). We now define:

$$A/\mathcal{I} := \text{colim}\{A \rightarrow A/tA\}_{t \in \mathcal{I}} \quad (3.27)$$

the colimit in (3.27) being taken in the category $\text{Comm}(\mathbf{C})$. By definition, it follows that A/\mathcal{I} is a commutative monoid in $(\mathbf{C}, \otimes, 1)$ and that the canonical morphism $A \rightarrow A/\mathcal{I}$ is a morphism of commutative monoids.

Lemma 3.8. Let A be a commutative monoid object in $(\mathbf{C}, \otimes, 1)$ and let $t \in \mathcal{E}(A)$. Then, the canonical map $A \rightarrow A/tA$ is an epimorphism of monoids, i.e., for any monoid B , the induced morphism

$$\text{Hom}_{\text{Comm}(\mathbf{C})}(A/tA, B) \rightarrow \text{Hom}_{\text{Comm}(\mathbf{C})}(A, B) \quad (3.28)$$

is an injection.

Proof. Let us denote by p the canonical morphism $p : A \rightarrow A/tA = \text{colim}(A \xleftarrow{t} A \rightarrow 0)$. Hence, $p \circ t = 0$. Let $f, g : A/tA \rightarrow B$ be morphisms of monoids such that $f \circ p = g \circ p$. Then, the morphism $f \circ p = g \circ p : A \rightarrow B$ makes B an A -algebra and f, g are maps of A -modules. Then, we have the following commutative diagram in $A\text{-Mod}$:

$$\begin{array}{ccccc} A & \xleftarrow{t} & A & \longrightarrow & 0 \\ \downarrow f \circ p = g \circ p & & \downarrow f \circ p = g \circ p & & \downarrow \\ B & \xleftarrow{1} & B & \xrightarrow{1} & B \end{array} \quad (3.29)$$

It follows that the morphism $f \circ p = g \circ p : A \longrightarrow B$ in $A - \text{Mod}$ must factor uniquely through the colimit $A/tA = \text{colim}(A \xleftarrow{t} A \longrightarrow 0)$. Hence, $f = g$ and the result follows. \square

Proposition 3.9. (a) Let A be a commutative monoid object in $(\mathbf{C}, \otimes, 1)$ and let $\mathcal{J} = (t)$ be a principal ideal in $\mathcal{E}(A)$ generated by a given $t \in \mathcal{E}(A)$. Then, $A/tA \cong A/\mathcal{J}$.

(b) Let A be a monoid object in $(\mathbf{C}, \otimes, 1)$ and let $\mathcal{J} \subseteq \mathcal{E}(A)$ be a given ideal. Then, the canonical morphism $A \longrightarrow A/\mathcal{J}$ is an epimorphism of monoids, i.e., for any monoid B , the induced morphism

$$\text{Hom}_{\text{Comm}(\mathbf{C})}(A/\mathcal{J}, B) \longrightarrow \text{Hom}_{\text{Comm}(\mathbf{C})}(A, B) \quad (3.30)$$

is an injection.

Proof. (a) By definition, we know that A/\mathcal{J} is given by the colimit

$$A/\mathcal{J} := \text{colim}\{p_x : A \longrightarrow A/xA\}_{x \in \mathcal{J}} \quad (3.31)$$

in $\text{Comm}(\mathbf{C})$. Suppose B is a monoid such that there are morphisms $q_x : A/xA \longrightarrow B$ of monoids such that $q_x \circ p_x = q_y \circ p_y$ for all $x, y \in \mathcal{J}$. For any element x in the principal ideal \mathcal{J} , there exists a natural morphism $p_{x/t} : A/xA \longrightarrow A/tA$ of monoids such that $p_t = p_{x/t} \circ p_x$. Then, we note that:

$$q_x \circ p_x = q_t \circ p_t = q_t \circ p_{x/t} \circ p_x \quad (3.32)$$

From Lemma 3.8, it now follows that $q_x = q_t \circ p_{x/t}$, i.e., each of the morphisms q_x factors through q_t . It follows that A/tA is the colimit $\text{colim}\{p_x : A \longrightarrow A/xA\}_{x \in \mathcal{J}} = A/\mathcal{J}$ in $\text{Comm}(\mathbf{C})$.

(b) Let $p : A \longrightarrow A/\mathcal{J}$ be the canonical morphism and let $f, g : A/\mathcal{J} \longrightarrow B$ be morphisms of monoids such that $f \circ p = g \circ p$. For each $t \in \mathcal{J}$, the morphism $p : A \longrightarrow A/\mathcal{J}$ factors through the canonical morphism $p_t : A \longrightarrow A/tA$ and let p'_t denote the canonical morphism $p'_t : A/tA \longrightarrow A/\mathcal{J}$ to the colimit A/\mathcal{J} . We note that

$$(f \circ p'_t) \circ p_t = f \circ p = g \circ p = (g \circ p'_t) \circ p_t \quad (3.33)$$

From Lemma 3.8, it now follows that $f \circ p'_t = g \circ p'_t$. Hence, the collection of morphisms $\{f \circ p'_t = g \circ p'_t\}_{t \in \mathcal{J}}$ factors uniquely through the colimit A/\mathcal{J} . This proves the result. \square

Finally, we show that if A is a Noetherian monoid and $\mathcal{J} \in \mathcal{E}(A)$ is an ideal in \mathcal{A} , the monoid A/\mathcal{J} is also Noetherian. We start with the following result.

Proposition 3.10. Let A be a Noetherian commutative monoid and let $t \in \mathcal{E}(A)$. Then, A/tA is a Noetherian monoid.

Proof. We consider a subobject $J \longrightarrow A/tA$ in $A/tA - \text{Mod}$. We then form the following pullback diagram in $A - \text{Mod}$:

$$\begin{array}{ccc} I & \longrightarrow & A \\ p' \downarrow & & p \downarrow \\ J & \longrightarrow & A/tA \end{array} \quad (3.34)$$

It is clear that I is a subobject of A in $A - Mod$ and hence finitely generated. By definition, we know that

$$A/tA := \text{colim}(A \xleftarrow{t} A \longrightarrow 0) \quad (3.35)$$

Since the morphism $A \longrightarrow 0$ is an epimorphism in $A - Mod$, it follows that the canonical morphism $p : A \longrightarrow A/tA$ is an epimorphism in $A - Mod$. Since $A - Mod$ is an abelian category, we know that epimorphisms are stable under pullback and hence the morphism $p' : I \longrightarrow J$ in (3.34) is an epimorphism in $A - Mod$. Hence, $\text{Im}(p') = J$. Now, we set

$$K := \text{Ker}(p') = \text{lim}(I \xrightarrow{p'} J \longleftarrow 0) \quad (3.36)$$

and consider the coimage

$$\text{Coim}(p') := \text{colim}(I \xleftarrow{i} K \longrightarrow 0) \quad (3.37)$$

Since $A - Mod$ is an abelian category, the image and the coimage of p' coincide and we have

$$J = \text{Im}(p') = \text{Coim}(p') = \text{colim}(I \xleftarrow{i} K \longrightarrow 0) \quad (3.38)$$

Next, we suppose that we have a filtered inductive system of objects $\{M_l\}$, $l \in L$ connected by monomorphisms in $A/tA - Mod$ and set $M := \text{colim}_{l \in L} M_l$. Let $h_l : M_l \longrightarrow M$, $l \in L$ denote the canonical morphisms. It is clear that we have a canonical morphism

$$\text{colim}_{l \in L} \text{Hom}_{A/tA - Mod}(J, M_l) \longrightarrow \text{Hom}_{A/tA - Mod}(J, M) \quad (3.39)$$

We now choose any morphism $f : J \longrightarrow M$ in $A/tA - Mod$. Let $f_I : I \longrightarrow J$ be the canonical morphism in $A - Mod$ induced by (3.38). We consider the morphism $f \circ f_I : I \longrightarrow M$ in $A - Mod$. Since I is finitely generated in $A - Mod$ and the system L is filtered, there exists some $l_0 \in L$ such that the morphism $f \circ f_I$ factors through M_{l_0} , i.e., there exists $g : I \longrightarrow M_{l_0}$ such that $h_{l_0} \circ g = f \circ f_I$. Further, we have

$$h_{l_0} \circ g \circ i = f \circ f_I \circ i = 0 : K \longrightarrow M \quad (3.40)$$

Since $h_{l_0} : M_{l_0} \longrightarrow M$ is a monomorphism, it follows from (3.40) that $g \circ i = 0 : K \longrightarrow M_{l_0}$. Since J is equal to the colimit in (3.38), it follows that the morphism $f : J \longrightarrow M$ factors through M_{l_0} in $A - Mod$.

Finally, since the canonical morphism $p : A \longrightarrow A/tA$ is also an epimorphism in the category of monoids (as shown in Proposition 3.9), it follows from Lemma 3.2 that

$$J \otimes_A A/tA \cong (J \otimes_{A/tA} A/tA) \otimes_A A/tA \cong J \otimes_{A/tA} (A/tA \otimes_A A/tA) \cong J \otimes_{A/tA} A/tA \cong J \quad (3.41)$$

Hence, for any object N in $A/tA - Mod$,

$$\text{Hom}_{A - Mod}(J, N) \cong \text{Hom}_{A/tA - Mod}(J \otimes_A A/tA, N) \cong \text{Hom}_{A/tA - Mod}(J, N) \quad (3.42)$$

Now, it follows that since the morphism $f : J \longrightarrow M$ factors through M_{l_0} in $A - Mod$, it actually factors through M_{l_0} in $A/tA - Mod$. Hence, J is finitely generated in $A/tA - Mod$. This proves the result. \square

Proposition 3.11. *Let A be a Noetherian commutative monoid and let $\mathcal{J} \subseteq \mathcal{E}(A)$ be an ideal in $\mathcal{E}(A)$. Then, A/\mathcal{J} is a Noetherian monoid.*

Proof. Since A is a Noetherian monoid, it follows from Proposition 3.4 that $\mathcal{E}(A)$ is actually a Noetherian ring. Hence, we may suppose that the ideal \mathcal{J} is generated by a finite set $\{t_1, \dots, t_k\} \subseteq \mathcal{E}(A)$. As in (2.20), we set, for $2 \leq i \leq k$:

$$A/(t_1, \dots, t_i)A := \operatorname{colim}(A/(t_1, \dots, t_{i-1})A \xleftarrow{t_i} A/(t_1, \dots, t_{i-1})A \longrightarrow 0) \quad (3.43)$$

From Proposition 3.10, we know that A/t_1A is Noetherian. From the recursive definition in (3.43), it follows that each $A/(t_1, \dots, t_i)A$ is Noetherian. Further, from (3.12), we know that

$$A/t_1A \otimes_A A/t_2A \otimes_A \cdots \otimes_A A/t_kA \cong A/(t_1, t_2, \dots, t_k)A \quad (3.44)$$

As in (3.2) in the proof of Lemma 3.2, for the finite collection $A/t_1A, A/t_2A, \dots, A/t_kA$ of A -algebras, we know that

$$C := \operatorname{Colim}_{1 \leq i \leq k} \{A \longrightarrow A/t_iA\} \cong A/t_1A \otimes_A A/t_2A \otimes_A \cdots \otimes_A A/t_kA \quad (3.45)$$

where the colimit in (3.45) is taken in the category of monoids. For any $1 \leq i \leq k$, we let $e_i : A/t_iA \longrightarrow C$ be the canonical morphism from A/t_iA to the colimit C described in (3.45). The induced morphism from A to C will be denoted by e . Further, for any $t \in \mathcal{J}$, let $p_t : A \longrightarrow A/tA$ denote the canonical epimorphism described in Lemma 3.8.

Since \mathcal{J} is generated by $\{t_1, \dots, t_k\}$, for any $t \in \mathcal{J}$, we can choose $s_i \in \mathcal{E}(A)$, $1 \leq i \leq k$ such that $t = \sum_{i=1}^k t_i s_i$. We note that:

$$e \circ t = e \circ \sum_{i=1}^k t_i s_i = \sum_{i=1}^k e \circ t_i s_i = \sum_{i=1}^k e_i \circ (p_{t_i} \circ t_i) \circ s_i = 0 \quad (3.46)$$

Hence, there exists a unique morphism $e_t : A/tA \longrightarrow C$, $e_t \circ p_t = e$, in $A - \operatorname{Mod}$ from the colimit $A/tA := \operatorname{colim}\{A \xleftarrow{t} A \longrightarrow 0\}$ to C that may be easily shown to be a morphism of monoids. Hence, the morphism $e : A \longrightarrow C$ factors through A/tA for any $t \in \mathcal{J}$ in the category of monoids. It follows that

$$A/\mathcal{J} = \operatorname{colim}_{t \in \mathcal{J}} \{A \longrightarrow A/tA\} \cong \operatorname{colim}_{1 \leq i \leq k} \{A \longrightarrow A/t_iA\} \cong A/t_1A \otimes_A A/t_2A \otimes_A \cdots \otimes_A A/t_kA \quad (3.47)$$

Combining (3.47) with (3.44) and the fact that $A/(t_1, t_2, \dots, t_k)A$ is Noetherian, it follows that A/\mathcal{J} is Noetherian. \square

Remark 3.12. The functor $\mathcal{E} : \operatorname{Comm}(\mathbf{C}) \longrightarrow \operatorname{Rings}$ plays a key role in our constructions above and in the rest of the paper. As such, we conclude this section by briefly summarizing some of the properties of this functor \mathcal{E} that we have proved above:

- (a) The functor \mathcal{E} preserves localizations, i.e., if $A \in \operatorname{Comm}(\mathbf{C})$ and if $S \subseteq \mathcal{E}(A)$ is a multiplicatively closed set, then $\mathcal{E}(A_S) = \mathcal{E}(A)_S$.
- (b) If $\{A_i\}_{i \in I}$ is a filtered system of objects in $\operatorname{Comm}(\mathbf{C})$, we have $\mathcal{E}(\operatorname{colim}_{i \in I} A_i) = \operatorname{colim}_{i \in I} \mathcal{E}(A_i)$.
- (c) If $A \in \operatorname{Comm}(\mathbf{C})$ is a Noetherian commutative monoid object, $\mathcal{E}(A)$ is a Noetherian ring.
- (d) If $\mathcal{J} \subseteq \mathcal{E}(A)$ is a finitely generated ideal, we have $\mathcal{E}(A/\mathcal{J}) = \mathcal{E}(A)/\mathcal{J}$.

4 Integral schemes and function field

In this section, we will introduce the definition and describe the properties of integral schemes over $(\mathbf{C}, \otimes, 1)$, in addition to reduced and irreducible schemes. In particular, we will show that an integral scheme is both reduced and irreducible. For our purposes, we will need to consider a second notion of integrality for monoids (see Definition 4.1) that we shall refer to as “weak integrality”. We will show that a reduced and irreducible scheme over $(\mathbf{C}, \otimes, 1)$ is weakly integral. Moreover, for any integral scheme X over $(\mathbf{C}, \otimes, 1)$, we will construct a field $k(X)$ that is the appropriate analogue of the function field of an ordinary integral scheme and show that this association is functorial with respect to dominant morphisms. Further, we will verify that the field $k(X)$ associated to an integral scheme X over $(\mathbf{C}, \otimes, 1)$ is completely determined by any open subscheme U of X . We mention here that when A is “weakly integral” in the sense of Definition 4.1 below and also “Noetherian” in the sense of [6] (which is different from the notion of Noetherian in Definition 3.1), we have constructed in [6] a monoid object $K(A) \in \mathbf{C}$ with some “field like properties”. We start by presenting the following two definitions.

Definition 4.1. (*Weakly integral monoids*) Let A be a commutative monoid object in $(\mathbf{C}, \otimes, 1)$. Then, we will say that A is weakly integral if $\mathcal{E}(A)$ is an integral domain.

Definition 4.2. (*Integral monoids*) Let A be a weakly integral monoid object in $(\mathbf{C}, \otimes, 1)$. We will say that A is an integral monoid if it satisfies the following two conditions:

- (1) For any element $s \in \mathcal{E}(A) = \text{Hom}_{A-\text{Mod}}(A, A)$ such that $s \neq 0$, the morphism $s : A \longrightarrow A$ is a monomorphism in $A - \text{Mod}$.
- (2) Let $S \subseteq \mathcal{E}(A)$ be the multiplicatively closed subset of all non-zero elements in the integral domain $\mathcal{E}(A)$ and let $K := A_S$. Then, K has no proper subobjects in $K - \text{Mod}$, i.e., any monomorphism $J \longrightarrow K$ in $K - \text{Mod}$ with $J \neq 0$ is an isomorphism.

Proposition 4.3. Let A be an integral monoid object of $(\mathbf{C}, \otimes, 1)$ and let $S \subseteq \mathcal{E}(A)$ be a multiplicatively closed subset such that $0 \notin S$. Then, the canonical morphism $i_S : A \longrightarrow A_S$ is a monomorphism in $A - \text{Mod}$.

Proof. Since A is an integral monoid, we know that for any $s \in S \subseteq \mathcal{E}(A)$, the morphism $s : A \longrightarrow A$ (and hence any $s^i : A \longrightarrow A$) is a monomorphism in $A - \text{Mod}$. We now have the following morphism of filtered inductive systems in $A - \text{Mod}$:

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \xrightarrow{1} & A & \xrightarrow{1} & \dots \\ 1 \downarrow & & s \downarrow & & s^2 \downarrow & & \\ A & \xrightarrow{s} & A & \xrightarrow{s} & A & \xrightarrow{s} & \dots \end{array} \quad (4.1)$$

with each vertical map in (4.1) a monomorphism. Hence, the induced morphism on filtered colimits $i_s : A \longrightarrow A_s$ of the horizontal rows in (4.1) is also a monomorphism. Again, it follows that the filtered colimit of monomorphisms $i_s : A \longrightarrow A_s$, $s \in S$,

$$i_S : A \longrightarrow A_S = \text{colim}_{s \in S} A_s \quad (4.2)$$

is a monomorphism. □

By definition, we know that an integral monoid A is also weakly integral. We will now show that if A is an integral monoid and $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is a Zariski open immersion of affine schemes, the monoid B is weakly integral.

Proposition 4.4. *Let A be an integral monoid object in $(\mathbf{C}, \otimes, 1)$ and let $f : A \rightarrow B$ be a morphism of commutative monoids inducing a Zariski open immersion of affine schemes. Then, B is a weakly integral monoid.*

Proof. For the integral monoid A , we let $S \subseteq \mathcal{E}(A)$ be the multiplicatively closed set of all non zero elements in $\mathcal{E}(A)$ and let $K := A_S$. Then, since A is integral, it follows from Definition 4.2 that K has no nonzero proper subobjects in $K - \text{Mod}$.

We now consider the following pushout square in the category of monoids:

$$\begin{array}{ccc} A & \xrightarrow{i_A} & K \\ f \downarrow & & \downarrow f_K \\ B & \xrightarrow{i_B} & L = K \otimes_A B \end{array} \quad (4.3)$$

Since $f : A \rightarrow B$ induces a Zariski open immersion, it follows that $f_K : K \rightarrow L$ also induces a Zariski open immersion. We choose any $t \in \mathcal{E}(L)$ and consider $\text{Ker}(t)$, which is a subobject of L in $L - \text{Mod}$. However, from the proof of Proposition 3.3, we know that since $f_K : K \rightarrow L$ induces a Zariski open immersion, every subobject of L in $L - \text{Mod}$ is extended from a subobject of K in $K - \text{Mod}$. Since K has no nonzero proper subobjects in $K - \text{Mod}$, it follows that $\text{Ker}(t) = 0$ or $\text{Ker}(t) = L$, i.e., if $t \neq 0$, then $\text{Ker}(t) = 0$. Now, suppose that there exists an element $t' \in \mathcal{E}(L)$ such that $t \circ t' = 0$. Then, if $t \neq 0$,

$$t \circ t' = 0 \quad \Rightarrow \quad \text{Im}(t') \subseteq \text{Ker}(t) = 0 \quad (4.4)$$

where we note that the image $\text{Im}(t')$ exists as a subobject of L in the abelian category $L - \text{Mod}$. It follows from (4.4) that $t' = 0$, i.e., $\mathcal{E}(L)$ is an integral domain.

Further, it follows from Proposition 4.3 that the canonical morphism $i_A : A \rightarrow K = A_S$ is a monomorphism in $A - \text{Mod}$. Since B is a flat A -module, it follows that $i_B = i_A \otimes_A B : B \rightarrow K \otimes_A B = L$ is a monomorphism in $B - \text{Mod}$. Finally, suppose that $s, s' \in \mathcal{E}(B)$ are morphisms such that $s \circ s' = 0$. Consider the extensions $s \otimes_B L, s' \otimes_B L$ of s, s' resp. to $\mathcal{E}(L)$. Since $\mathcal{E}(L)$ is an integral domain, it follows that

$$(s \otimes_B L) \circ (s' \otimes_B L) = ((s \circ s') \otimes_B L) = 0 \quad \Rightarrow \quad (s \otimes_B L) = 0 \text{ or } (s' \otimes_B L) = 0 \quad (4.5)$$

For sake of definiteness, we assume $s \otimes_B L = 0$. Then, we note that, under the isomorphism $\text{Hom}_{B-\text{Mod}}(B, L) \cong \text{Hom}_{L-\text{Mod}}(L, L)$, the morphism $s \otimes_B L \in \text{Hom}_{L-\text{Mod}}(L, L)$ corresponds to

$$(s \otimes_B L) \circ i_B = i_B \circ s : B \xrightarrow{s} B \xrightarrow{i_B} L \quad (4.6)$$

in $\text{Hom}_{B-\text{Mod}}(B, L)$. Hence, $i_B \circ s = 0$. Since i_B is a monomorphism in $B - \text{Mod}$, it follows that $s : B \rightarrow B$ is zero. Hence, $\mathcal{E}(B)$ is an integral domain and B is a weakly integral monoid. □

Definition 4.5. Let X be a scheme over $(\mathbf{C}, \otimes, 1)$. We will say that X is an integral scheme if, given any Zariski open immersion $Y \rightarrow X$ with $Y = \text{Spec}(A)$ affine, A is an integral monoid in $(\mathbf{C}, \otimes, 1)$.

We will say that X is weakly integral (resp. reduced) if given any Zariski open immersion $Y \rightarrow X$ with $Y = \text{Spec}(A)$ affine, $\mathcal{E}(A)$ is an integral domain (resp. a reduced ring).

We will say that a scheme X is irreducible if, given any Zariski open immersions, $U \rightarrow X$, $V \rightarrow X$ with U and V nontrivial, the fibre product $U \times_X V$ is a nontrivial.

Proposition 4.6. Let X be a scheme over $(\mathbf{C}, \otimes, 1)$ that is both reduced and irreducible. Then, X is a weakly integral scheme.

Proof. Let $i : U \rightarrow X$ be a Zariski open immersion with $U = \text{Spec}(A)$ affine. Suppose that we choose $s, t \in \mathcal{E}(A)$ such that $s \neq 0$ and $t \neq 0$. From Corollary 2.8, it follows that $\mathcal{E}(A_s) = \mathcal{E}(A)_s$. Since $\mathcal{E}(A)$ is reduced, $\mathcal{E}(A)_s \neq 0$ and hence $A_s \neq 0$. Similarly, $A_t \neq 0$.

From Proposition 2.5, we know that the compositions $\text{Spec}(A_s) \xrightarrow{i_t} \text{Spec}(A) \xrightarrow{i} X$ and $\text{Spec}(A_t) \xrightarrow{i_s} \text{Spec}(A) \xrightarrow{i} X$ are Zariski immersions. Since X is irreducible and both $\text{Spec}(A_s)$ and $\text{Spec}(A_t)$ are non trivial, it follows that the fibre product $\text{Spec}(A_s) \times_X \text{Spec}(A_t)$ is non trivial.

Further, since a Zariski open immersion is a monomorphism in the category of schemes over $(\mathbf{C}, \otimes, 1)$, it follows that, given morphisms $f_s : Y \rightarrow \text{Spec}(A_s)$, $f_t : Y \rightarrow \text{Spec}(A_t)$ of schemes such that $i \circ (i_s \circ f_s) = i \circ (i_t \circ f_t)$, we must have $i_s \circ f_s = i_t \circ f_t$. Hence, we have an isomorphism of fibre products:

$$\text{Spec}(A_s) \times_X \text{Spec}(A_t) \xrightarrow{\cong} \text{Spec}(A_s) \times_{\text{Spec}(A)} \text{Spec}(A_t) = \text{Spec}(A_{st}) = \text{Spec}(A_s \otimes_A A_t) \quad (4.7)$$

It follows that $\text{Spec}(A_{st})$ is non trivial. Hence, $A_{st} \neq 0$ and therefore $st \neq 0$. Hence, $\mathcal{E}(A)$ is an integral domain and A is a weakly integral monoid. This proves the result. \square

We will now prove a partial converse to Proposition 4.6. From Definition 4.5, it is clear that an integral scheme is always reduced. We start by showing that if A is an integral monoid, then $\text{Spec}(A)$ is irreducible.

Proposition 4.7. Let A be an integral monoid object in $(\mathbf{C}, \otimes, 1)$. Then, $\text{Spec}(A)$ is an irreducible scheme.

Proof. It suffices to show that if $U \rightarrow \text{Spec}(A)$, $V \rightarrow \text{Spec}(A)$ are Zariski open immersions from affine schemes U and V , the fibre product $U \times_{\text{Spec}(A)} V$ is non trivial. Suppose, therefore that $f : A \rightarrow B$, $g : A \rightarrow C$, $B \neq 0$, $C \neq 0$, are morphisms of monoids inducing Zariski open immersions of affine schemes such that $B \otimes_A C = 0$. We let $S \subseteq \mathcal{E}(A)$ be the multiplicatively closed set of all nonzero elements of $\mathcal{E}(A)$ and we set $K := A_S$. We now consider the following pushout squares in $\text{Comm}(\mathbf{C})$:

$$\begin{array}{ccc} A & \xrightarrow{i} & K \\ f \downarrow & & f_K \downarrow \\ B & \xrightarrow{i_B} & B_K = B \otimes_A K \end{array} \quad \begin{array}{ccc} A & \xrightarrow{i} & K \\ g \downarrow & & g_K \downarrow \\ C & \xrightarrow{i_C} & C_K = C \otimes_A K \end{array} \quad (4.8)$$

Since A is an integral monoid, it follows as in the proof of Proposition 4.3 that any morphism $s \in \text{Hom}_{A-\text{Mod}}(A, A) = \mathcal{E}(A)$, $s \neq 0$ is a monomorphism in $A - \text{Mod}$. Since B is flat over A , it follows that

$$\mathcal{E}(f)(s) = s \otimes_A B : B = A \otimes_A B \xrightarrow{s \otimes_A B} A \otimes_A B = B \quad (4.9)$$

is a monomorphism in $B - \text{Mod}$. Hence, $\mathcal{E}(f)(s) \neq 0$ and therefore $\mathcal{E}(f)(S) \subseteq \mathcal{E}(B)$ is a multiplicatively closed subset of $\mathcal{E}(B)$ containing 1_B and not containing 0 . It follows from the definition of localisation in (2.3) that

$$B_K = B \otimes_A K = B_{\mathcal{E}(f)(S)} \quad (4.10)$$

Since A is an integral monoid, it follows from Proposition 4.3 that the canonical morphism $i : A \rightarrow K = A_S$ is a monomorphism. Since $f : A \rightarrow B$ induces a Zariski open immersion, B is a flat A -module. Consequently, $i_B = i \otimes_A B : A \otimes_A B = B \rightarrow K \otimes_A B = B_K = B_{\mathcal{E}(f)(S)}$ is a monomorphism in $B - \text{Mod}$ and hence $B_K \neq 0$. Similarly, $C_K \neq 0$.

Now, suppose that $f_K = 0$. Then, $i_B \circ f = f_K \circ i = 0$. Hence,

$$0 = (i_B \circ f) \otimes_A B : B \xrightarrow{f \otimes_A B} B \otimes_A B = B \xrightarrow{i_B \otimes_A B} B_K \otimes_A B = B \otimes_A B \otimes_A K = B_K \quad (4.11)$$

where the equality $B \otimes_A B = B$ in (4.11) follows from Lemma 3.2. From (4.11), we have $0 = (i_B \circ f) \otimes_A B = i_B : B \rightarrow B_K$ which contradicts the fact that i_B is a monomorphism in $B - \text{Mod}$. Hence, we must have $f_K \neq 0$.

We now consider the following kernels:

$$T_A := \lim_{A-\text{Mod}}(K \xrightarrow{f_K} B_K \leftarrow 0) \quad T_K := \lim_{K-\text{Mod}}(K \xrightarrow{f_K} B_K \leftarrow 0) \quad (4.12)$$

where the first limit in (4.12) is taken in the category $A - \text{Mod}$ and the second is taken in $K - \text{Mod}$. Since A is an integral monoid, we know that K has no non-zero proper subobjects in $K - \text{Mod}$. Then, since $f_K \neq 0$ as shown above, the subobject T_K of K in $K - \text{Mod}$ must be zero.

Further, as mentioned in Section 2, the canonical morphism $i : A \rightarrow A_S = K$ defined by the localisation is an epimorphism in $\text{Comm}(\mathbf{C})$. Using Lemma 3.2, we have $K \otimes_A K = K$. Since $K = A_S$ is a flat A -module, it now follows that

$$T_A \otimes_A K = \lim_{K-\text{Mod}}(K \otimes_A K = K \xrightarrow{f_K} B_K \otimes_A K = B \otimes_A K \otimes_A K = B_K \leftarrow 0) = T_K = 0 \quad (4.13)$$

However, we also know that

$$\text{Hom}_{A-\text{Mod}}(T_A, K) \cong \text{Hom}_{K-\text{Mod}}(T_A \otimes_A K, K) = \text{Hom}_{K-\text{Mod}}(T_K, K) = 0 \quad (4.14)$$

Since T_A is a subobject of K in $A - \text{Mod}$, it follows from (4.14) that $T_A = 0$. Combining with (4.12), it follows that $f_K : K \rightarrow B_K$ is a monomorphism in $A - \text{Mod}$. Since C is a flat A -module, it follows that

$$f_K \otimes_A C : C_K = K \otimes_A C \rightarrow B_K \otimes_A C = (B \otimes_A C) \otimes_A K = 0 \quad (4.15)$$

is a monomorphism in $C - \text{Mod}$. Hence, $C_K = 0$, which is a contradiction. \square

Proposition 4.8. *Let X be an integral scheme over $(\mathbf{C}, \otimes, 1)$. Then, X is reduced and irreducible.*

Proof. Our argument is similar to the proof of [6, Proposition 2.11]. It is clear that an integral scheme X is also reduced. Suppose that X is not irreducible, i.e., there exist non-trivial Zariski open immersions $U = \text{Spec}(A) \rightarrow X$ and $V = \text{Spec}(B) \rightarrow X$ such that $U \times_X V$ is trivial. We consider the induced morphism $p : U \amalg V \rightarrow X$.

Then, if $W = \text{Spec}(C) \rightarrow X$ is any other Zariski open immersion, it follows that

$$(U \times_X W) \times_W (V \times_X W) = (U \times_X V) \times_X W \quad (4.16)$$

is trivial. Since C is an integral monoid, it follows from Proposition 4.7 that $W = \text{Spec}(C)$ is also irreducible. Hence, at least one of $U \times_X W$ and $V \times_X W$ must be trivial. We now construct the following pullback square

$$\begin{array}{ccc} (U \times_X W) \amalg (V \times_X W) & \xrightarrow{p_W} & W \\ \downarrow & & \downarrow \\ U \amalg V & \xrightarrow{p} & X \end{array} \quad (4.17)$$

Since at least one of $U \times_X W$ and $V \times_X W$ is trivial, it follows that $p_W : (U \times_X W) \amalg (V \times_X W) \rightarrow W$ is equal to at least one of the Zariski open immersions $U \times_X W \rightarrow W$ or $V \times_X W \rightarrow W$. Hence, for any Zariski open immersion $W = \text{Spec}(C) \rightarrow X$, the morphism p_W obtained from the pullback square (4.17) is always a Zariski open immersion. It follows that $p : U \amalg V = \text{Spec}(A \oplus B) \rightarrow X$ is a Zariski open immersion. Since X is integral, $\mathcal{E}(A \oplus B)$ must be an integral domain. However, if we consider the canonical morphisms

$$e_A : A \oplus B \rightarrow A \rightarrow A \oplus B \quad e_B : A \oplus B \rightarrow B \rightarrow A \oplus B \quad (4.18)$$

in $\mathcal{E}(A \oplus B)$, it is clear that $e_A \circ e_B = 0$. Hence, at least one of A and B is zero. This proves that X is irreducible. \square

Let X be an integral scheme over $(\mathbf{C}, \otimes, 1)$. We will now construct the analogue of the usual function field of X . Consider the collection of pairs (U, t_U) such that $U = \text{Spec}(A) \rightarrow X$ is a Zariski open immersion, $A \neq 0$ and $t_U \in \mathcal{E}(A)$. Given non-trivial Zariski open immersions $U = \text{Spec}(A) \rightarrow X$ and $V = \text{Spec}(B) \rightarrow X$, we will say that two pairs (U, t_U) and (V, t_V) are equivalent, written $(U, t_U) \sim (V, t_V)$, if there exists a Zariski immersion $W = \text{Spec}(C) \rightarrow U \times_X V$ such that the restrictions of $t_U \in \mathcal{E}(A)$ and $t_V \in \mathcal{E}(B)$ to $\mathcal{E}(C)$ are equal. Since X is irreducible, the collection of these equivalence classes defines an ordinary unital commutative ring, which we denote by $k(X)$.

Proposition 4.9. *Let X be an integral scheme over $(\mathbf{C}, \otimes, 1)$. Then, $k(X)$ is a field. Further, let $f : Y \rightarrow X$ be a dominant morphism of integral schemes, i.e., for any open immersion $V = \text{Spec}(B) \rightarrow X$ with $B \neq 0$, the fibre product $Y \times_X V$ is non-trivial. Then, f induces a morphism $k(f) : k(X) \rightarrow k(Y)$ of fields.*

Proof. Let us consider a pair (U, t_U) defining a class in $k(X)$ with $U = \text{Spec}(A)$, $t_U \in \mathcal{E}(A)$, $t_U \neq 0$. Since $\mathcal{E}(A)$ is an integral domain, $A_{t_U} \neq 0$ as in the proof of Proposition 4.6. From Corollary 2.8, $\mathcal{E}(A_{t_U}) = \mathcal{E}(A)_{t_U}$ and hence we can consider the pair $(\text{Spec}(A_{t_U}), t_U^{-1})$ defining a class in $k(X)$. Then, as in (4.7),

$$\text{Spec}(A_{t_U}) \times_X \text{Spec}(A) \cong \text{Spec}(A_{t_U}) \times_{\text{Spec}(A)} \text{Spec}(A) \cong \text{Spec}(A_{t_U}) \quad (4.19)$$

and hence the product of the classes in $k(X)$ defined by (U, t_U) and $(\text{Spec}(A_U), t_U^{-1})$ is unity. Hence, $k(X)$ is a field.

Now suppose that $f : Y \rightarrow X$ is a dominant morphism of integral schemes as described above. We choose any pair (V, t_V) , $V = \text{Spec}(B)$, $t_V \in \mathcal{E}(B)$ defining a class in $k(X)$ and consider $U := V \times_X Y$. Since f is dominant, U is non-trivial. Hence, we can choose an affine scheme $U' = \text{Spec}(A)$, $A \neq 0$ admitting a Zariski open immersion $U' \rightarrow U$ into U . We consider the composition $U' = \text{Spec}(A) \rightarrow U \rightarrow V = \text{Spec}(B)$ and let $g : B \rightarrow A$ denote the corresponding morphism of monoids. We now associate the class defined by (V, t_V) in $k(X)$ to the class defined by $(U', \mathcal{E}(g)(t_V))$ in $k(Y)$. It is clear that this defines a morphism $k(f) : k(X) \rightarrow k(Y)$. \square

Henceforth, for any integral scheme X over $(\mathbf{C}, \otimes, 1)$, we will say that $k(X)$ is the function field of X . We will now prove that the function field of such a scheme is completely determined by any open subscheme.

Corollary 4.10. *Let X be an integral scheme and let $U \rightarrow X$ be a Zariski open immersion with U non-trivial. Then, we have an isomorphism of function fields $k(X) \cong k(U)$.*

Proof. We consider any pair (V, t_V) , $V = \text{Spec}(B)$, $t_V \in \mathcal{E}(B)$ defining a class in $k(X)$. Since X is irreducible, the Zariski immersion $U \rightarrow X$ is dominant. Hence, as in the proof of Proposition 4.9, the pair (V, t_V) determines a class in $k(U)$ via the induced morphism $k(X) \rightarrow k(U)$. Conversely, consider any pair (W, t_W) , $W = \text{Spec}(C)$, $t_W \in \mathcal{E}(C)$ defining a class in $k(U)$. Then, using the composition $W \rightarrow U \rightarrow X$ of Zariski open immersions, it follows that (W, t_W) defines a class in $k(X)$. It is easy to check that these associations are inverses of each other and we have an isomorphism $k(X) \cong k(U)$. \square

Let A be an integral monoid and let $S \subseteq \mathcal{E}(A)$ be the multiplicatively closed subset of all nonzero elements of $\mathcal{E}(A)$. Then, we will always denote the localisation A_S by $F(A)$. From Corollary 2.8, it follows that $\mathcal{E}(F(A)) = \mathcal{E}(A)_S$ and hence $\mathcal{E}(F(A))$ is the field of fractions of the integral domain $\mathcal{E}(A)$. From Corollary 4.10, it suffices to describe the function field for integral schemes that are affine. Therefore, let A be an integral monoid in $(\mathbf{C}, \otimes, 1)$ such that $\text{Spec}(A)$ is an integral scheme. We can now describe the function field of $\text{Spec}(A)$ more explicitly.

Proposition 4.11. *Let A be an integral monoid object in $(\mathbf{C}, \otimes, 1)$ such that $\text{Spec}(A)$ is an integral scheme. Then, the function field $k(\text{Spec}(A))$ can be described as the filtered colimit*

$$k(\text{Spec}(A)) \cong \text{colim}_B \mathcal{E}(F(B)) \quad (4.20)$$

where the colimit in (4.20) ranges over all Zariski open immersions $\text{Spec}(B) \rightarrow \text{Spec}(A)$ with $B \neq 0$.

Proof. We note that if $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is a Zariski open immersion and $\text{Spec}(A)$ is an integral scheme, it follows from Definition 4.5 that B is integral. If $B \neq 0$, then we can define $F(B)$ and there is an induced morphism $\mathcal{E}(F(A)) \rightarrow \mathcal{E}(F(B))$. Since $\text{Spec}(A)$ is irreducible, it follows that the colimit in (4.20) is filtered. We let C denote the colimit $C := \text{colim}_B \mathcal{E}(F(B))$.

We now consider a pair (U, t_U) with $U = \text{Spec}(B)$, $t_U \in \mathcal{E}(B)$ defining a class in $k(\text{Spec}(A))$. Then, $\text{Spec}(B)$ admits a Zariski open immersion into $\text{Spec}(A)$ and we associate (U, t_U) to the element of C defined by $t_U \in \mathcal{E}(B) \subseteq \mathcal{E}(F(B))$.

Conversely, suppose that we choose any element $t \in \mathcal{E}(F(B))$ with $\text{Spec}(B)$ admitting a Zariski open immersion into $\text{Spec}(A)$. Then, $\mathcal{E}(F(B))$ is the field of fractions of $\mathcal{E}(B)$ and hence we may express t as $t = t_1 t_2^{-1}$, with $t_1, t_2 \in \mathcal{E}(B)$. Since $\mathcal{E}(B_{t_2}) = \mathcal{E}(B)_{t_2} \neq 0$, we have $B_{t_2} \neq 0$. From Proposition 2.5, we know that $\text{Spec}(B_{t_2}) \rightarrow \text{Spec}(B)$ is a Zariski open immersion. We note that $t = t_1 t_2^{-1} \in \mathcal{E}(B)_{t_2} = \mathcal{E}(B_{t_2})$. Hence, we can associate the class in C defined by $t \in \mathcal{E}(F(B))$ to the class in $k(\text{Spec}(A))$ defined by the pair $(\text{Spec}(B_{t_2}), t_1 t_2^{-1})$. It is clear that these associations are inverse to each other and hence we have an isomorphism $k(\text{Spec}(A)) \cong \underset{B}{\text{colim}} \mathcal{E}(F(B))$ in (4.20). \square

5 Closed subschemes and quasi-coherent sheaves of algebras

Since the theory of schemes over $(\mathbf{C}, \otimes, 1)$ is developed by abstracting the properties of Zariski open immersions in usual algebraic geometry, open subschemes and open immersions fit naturally into this formalism. However, it is more difficult to develop an analogous notion of closed subschemes. This will be the purpose of this section. We start by showing that, if a scheme X is semi-separated, there is a one-one correspondence between quasi-coherent sheaves of algebras on the scheme X and the collection of affine morphisms $Y \rightarrow X$. In particular, we apply this to construct closed subschemes and the local ring corresponding to an integral subscheme of a Noetherian, integral and semi-separated scheme.

Definition 5.1. *A scheme X over $(\mathbf{C}, \otimes, 1)$ will be said to be semi-separated if, given Zariski open immersions $Y_1 \rightarrow X$, $Y_2 \rightarrow X$ with Y_1, Y_2 affine, the fibre product $Y_1 \times_X Y_2$ is also an affine scheme.*

Definition 5.2. *A morphism $f : Y \rightarrow X$ of schemes over $(\mathbf{C}, \otimes, 1)$ will be said to be affine if given a Zariski open immersion $U \rightarrow X$ with U affine, the fibre product $Y \times_X U$ is also affine.*

Definition 5.3. *Let X be a scheme over $(\mathbf{C}, \otimes, 1)$ and let $\text{ZarAff}(X)$ denote the category of Zariski open immersions $U \rightarrow X$ with U affine. Suppose that we have a functor:*

$$\mathcal{O} : \text{ZarAff}(X)^{\text{op}} \rightarrow \text{Comm}(\mathbf{C}) \quad (5.1)$$

For the sake of convenience, we will denote by $\mathcal{O}(U)$ the monoid associated to an object $U \rightarrow X$ in $\text{ZarAff}(X)$ by the functor \mathcal{O} . We will say that \mathcal{O} defines a quasi-coherent sheaf of algebras on X if \mathcal{O} satisfies the following conditions:

- (a) *For any object $U = \text{Spec}(A_U) \rightarrow X$ in $\text{ZarAff}(X)$, $\mathcal{O}(U)$ is an A_U -algebra.*
- (b) *Let $f : V = \text{Spec}(A_V) \rightarrow U = \text{Spec}(A_U)$ be a morphism in $\text{ZarAff}(X)$. Then, $\mathcal{O}(V) = \mathcal{O}(U) \otimes_{A_U} A_V$.*

In particular, the quasi-coherent sheaf of algebras on X defined by associating an object $U = \text{Spec}(A_U) \rightarrow X$ in $\text{ZarAff}(X)$ to $A_U \in \text{Comm}(\mathbf{C})$ will be referred to as the structure sheaf \mathcal{O}_X of the scheme X .

We remark here that for (not necessarily Noetherian) schemes over $(\mathbf{C}, \otimes, 1)$, we have studied structures on the derived category of quasi-coherent sheaves in [4].

Proposition 5.4. *Let X be a semi-separated scheme over $(\mathbf{C}, \otimes, 1)$. Then, there is a one-one correspondence between quasi-coherent sheaves of algebras on X and the collection of affine morphisms $Y \rightarrow X$.*

Proof. First, we consider an affine morphism $f : Y \rightarrow X$. Then, for any object $U = \text{Spec}(A_U) \rightarrow X$ in $\text{ZarAff}(X)$, $Y \times_X U$ must be affine and we let $Y \times_X U = \text{Spec}(B_U)$. Then, we can define a functor

$$\mathcal{O} : \text{ZarAff}(X)^{\text{op}} \rightarrow \text{Comm}(\mathbf{C}) \quad \mathcal{O}(U) := B_U \quad (5.2)$$

The induced morphism $\text{Spec}(B_U) = Y \times_X U \rightarrow U = \text{Spec}(A_U)$ ensures that B_U is an A_U -algebra. Further, suppose that $V = \text{Spec}(A_V) \rightarrow \text{Spec}(A_U) = U$ is a morphism in $\text{ZarAff}(X)$. Then, we have

$$\text{Spec}(B_V) := Y \times_X V = (Y \times_X U) \times_U V = \text{Spec}(B_U \otimes_{A_U} A_V) \quad (5.3)$$

and hence $\mathcal{O}(V) = B_V = B_U \otimes_{A_U} A_V = \mathcal{O}(U) \otimes_{A_U} A_V$. Hence, \mathcal{O} is a quasi-coherent sheaf of algebras on X .

Conversely, suppose that we are given a quasi-coherent sheaf \mathcal{O} of algebras on X . Let us choose an affine cover $\{U_i = \text{Spec}(A_i) \rightarrow X\}_{i \in I}$ of X . For each $i \in I$, we set $B_i = \mathcal{O}(U_i)$ and $V_i = \text{Spec}(B_i)$. Since X is semi-separated, the fibre products $U_i \times_X U_j$, $i, j \in I$ are all affine and we set $\text{Spec}(A_{i,j}) = U_i \times_X U_j$. Since \mathcal{O} is quasi-coherent, we know that

$$A_{i,j} \otimes_{A_i} B_i = A_{i,j} \otimes_{A_i} \mathcal{O}(U_i) = \mathcal{O}(U_i \times_X U_j) = A_{i,j} \otimes_{A_j} \mathcal{O}(U_j) = A_{i,j} \otimes_{A_j} B_j \quad (5.4)$$

Further, for any $i, j, k \in I$, we set

$$U_{ij} := U_i \times_X U_j \quad U_{ijk} = U_i \times_X U_j \times_X U_k \quad (5.5)$$

We now define

$$Y' := \coprod_{i \in I} V_i \quad R_{i,j} := \text{Spec}(\mathcal{O}(U_i \times_X U_j)) = \text{Spec}(A_{i,j} \otimes_{A_i} B_i) = \text{Spec}(A_{i,j} \otimes_{A_j} B_j) \quad (5.6)$$

for all $(i, j) \in I^2$. It is clear that the morphism

$$R_{i,j} = \text{Spec}(A_{i,j} \otimes_{A_i} B_i) = \text{Spec}(A_{i,j}) \times_{\text{Spec}(A_i)} \text{Spec}(B_i) \rightarrow \text{Spec}(A_i) \times_{\text{Spec}(A_i)} \text{Spec}(B_i) = \text{Spec}(B_i) = V_i \quad (5.7)$$

obtained by base change from $U_i \times_X U_j = \text{Spec}(A_{i,j}) \rightarrow U_i = \text{Spec}(A_i)$ is a Zariski open immersion. Moreover, for any $i \in I$, the morphism

$$R_{i,i} = \text{Spec}(A_{i,i} \otimes_{A_i} B_i) = \text{Spec}(A_i \otimes_{A_i} B_i) = \text{Spec}(B_i) = V_i \rightarrow \text{Spec}(B_i \otimes B_i) = V_i \times V_i \quad (5.8)$$

is identical to the “diagonal morphism” $V_i \longrightarrow V_i \times V_i$. Also, for any $i, j, k \in I$, we consider

$$\begin{aligned}
R_{i,j} \times_{V_j} R_{j,k} &= \text{Spec}(A_{i,j} \otimes_{A_j} B_j \otimes_{B_j} A_{j,k} \otimes_{A_k} B_k) \\
&= \text{Spec}(A_{i,j} \otimes_{A_j} A_{j,k} \otimes_{A_k} B_k) \\
&= \text{Spec}(A_{i,j}) \times_{\text{Spec}(A_j)} \text{Spec}(A_j) \times_X \text{Spec}(A_k) \times_{\text{Spec}(A_k)} \text{Spec}(B_k) \\
&= \text{Spec}(A_{i,j}) \times_X \text{Spec}(A_k) \times_{\text{Spec}(A_k)} \text{Spec}(B_k) \\
&= (\text{Spec}(A_i) \times_X \text{Spec}(A_j) \times_X \text{Spec}(A_k)) \times_{\text{Spec}(A_k)} \text{Spec}(B_k) \\
&= U_{ijk} \times_{U_k} V_k
\end{aligned} \tag{5.9}$$

Proceeding in a manner similar to (5.9), we can show that

$$R_{i,j} \times_{V_j} R_{j,k} = (\text{Spec}(A_i) \times_X \text{Spec}(A_j) \times_X \text{Spec}(A_k)) \times_{\text{Spec}(A_i)} \text{Spec}(B_i) = U_{ijk} \times_{U_i} V_i \tag{5.10}$$

From (5.6) and (5.7), we know that

$$\begin{aligned}
R_{i,k} &= (\text{Spec}(A_i) \times_X \text{Spec}(A_k)) \times_{\text{Spec}(A_i)} \text{Spec}(B_i) = U_{ik} \times_{U_i} V_i \\
&= (\text{Spec}(A_i) \times_X \text{Spec}(A_k)) \times_{\text{Spec}(A_k)} \text{Spec}(B_k) = U_{ik} \times_{U_k} V_k
\end{aligned} \tag{5.11}$$

Then, there exists a morphism $r_{ijk} : R_{i,j} \times_{V_j} R_{j,k} \longrightarrow R_{i,k}$ that may be described in either of the two following ways:

$$\begin{array}{ccc}
R_{i,j} \times_{V_j} R_{j,k} = U_{ijk} \times_{U_i} V_i & \xrightarrow{r_{ijk}} & U_{ik} \times_{U_i} V_i = R_{i,k} \\
= \downarrow & & = \downarrow \\
R_{i,j} \times_{V_j} R_{j,k} = U_{ijk} \times_{U_k} V_k & \xrightarrow{r_{ijk}} & U_{ik} \times_{U_k} V_k = R_{i,k}
\end{array} \tag{5.12}$$

Further, there is a natural morphism $(q_i, q_k) : R_{i,k} \longrightarrow V_i \times V_k$ induced by the pair of morphisms:

$$q_i : R_{i,k} = U_{ik} \times_{U_i} V_i \longrightarrow U_i \times_{U_i} V_i = V_i \quad q_k : R_{i,k} = U_{ik} \times_{U_k} V_k \longrightarrow U_k \times_{U_k} V_k = V_k \tag{5.13}$$

Similarly, there is a natural morphism $(p_i, p_k) : R_{i,j} \times_{V_j} R_{j,k} \longrightarrow V_i \times V_k$ induced by the pair of morphisms:

$$\begin{aligned}
p_i : R_{i,j} \times_{V_j} R_{j,k} &= U_{ijk} \times_{U_i} V_i \longrightarrow U_i \times_{U_i} V_i = V_i \\
p_k : R_{i,j} \times_{V_j} R_{j,k} &= U_{ijk} \times_{U_k} V_k \longrightarrow U_k \times_{U_k} V_k = V_k
\end{aligned} \tag{5.14}$$

From the top row of (5.12) it is clear that $q_i \circ r_{ijk} = p_i$ and from the bottom row of (5.12), it is clear that $q_k \circ r_{ijk} = p_k$. It follows that we have

$$(q_i, q_k) \circ r_{ijk} = (p_i, p_k) : R_{i,j} \times_{V_j} R_{j,k} \longrightarrow R_{i,k} \longrightarrow V_i \times V_k \tag{5.15}$$

Hence, the natural morphism $R_{i,j} \times_{V_j} R_{j,k} \longrightarrow V_i \times V_k$ factors through $R_{i,k}$. Finally, since $R_{i,j} = R_{j,i}$ $\forall i, j \in I$, it follows that

$$R = \coprod_{(i,j) \in I^2} R_{i,j} \subseteq Y' \times Y' \tag{5.16}$$

defines an equivalence relation on Y' satisfying the conditions of [17, Proposition 2.18]. Hence, from [17, Proposition 2.18], it follows that $Y := Y'/R$ defines a scheme Y equipped with a natural morphism $Y \longrightarrow X$.

We now need to show that the morphism $Y \longrightarrow X$ is affine. If $X = \text{Spec}(A)$ is already affine, then $Y = \text{Spec}(\mathcal{O}(X))$ and the result is clear. In general, we notice that given any Zariski open

immersion $W \rightarrow X$ with $W = \text{Spec}(C)$ affine, each of the operations used in constructing the scheme Y above commutes with the pullback to W . Hence, $W \times_X Y = \text{Spec}(\mathcal{O}(W))$ and the induced morphism $Y \rightarrow X$ is affine. Finally, it may be verified that the two associations defined above are inverses of each other. \square

Let X be a Noetherian scheme. We consider a “quasi-coherent sheaf of ideals” \mathcal{I} on X , i.e., to each object $U = \text{Spec}(A) \rightarrow X$ in $\text{ZarAff}(X)^{op}$, we associate a proper ideal $\mathcal{I}(U) \subseteq \mathcal{E}(A)$ such that given any Zariski open immersion $V = \text{Spec}(B) \rightarrow U = \text{Spec}(A)$,

$$\mathcal{I}(V) = \mathcal{I}(U) \otimes_{\mathcal{E}(A)} \mathcal{E}(B) \quad (5.17)$$

We will now show that given a quasi-coherent sheaf of ideals \mathcal{I} on X , we can associate a quasi-coherent sheaf of algebras $\mathcal{O}_X/\mathcal{I}$ on X in the sense of Definition 5.3.

Proposition 5.5. *Let A be a Noetherian monoid and let $f : A \rightarrow A'$ be a morphism of monoid objects in $(\mathbf{C}, \otimes, 1)$ inducing a Zariski open immersion of affine schemes. Let $\mathcal{I} \subseteq \mathcal{E}(A)$ be an ideal and let $\mathcal{I}' \subseteq \mathcal{E}(A')$ be the ideal in $\mathcal{E}(A')$ extended from \mathcal{I} using the induced morphism $\mathcal{E}(f) : \mathcal{E}(A) \rightarrow \mathcal{E}(A')$, i.e., $\mathcal{I}' = \mathcal{I} \otimes_{\mathcal{E}(A)} \mathcal{E}(A')$. Then, $A/\mathcal{I} \otimes_A A' \cong A'/\mathcal{I}'$.*

Proof. Since $f : A \rightarrow A'$ induces a Zariski open immersion, it follows from Proposition 3.3 that A' is also Noetherian. From Proposition 3.4, it follows that $\mathcal{E}(A)$ and $\mathcal{E}(A')$ are Noetherian rings. Hence, we may choose a finite set $\{t_1, \dots, t_k\}$ of generators for the ideal $\mathcal{I} \subseteq \mathcal{E}(A)$. Since \mathcal{I}' is extended from \mathcal{I} , it follows that $\mathcal{I}' \subseteq \mathcal{E}(A')$ is generated by $\{\mathcal{E}(f)(t_1), \dots, \mathcal{E}(f)(t_k)\}$. Then, as in (3.47), we know that

$$\begin{aligned} A/\mathcal{I} &\cong A/t_1 A \otimes_A A/t_2 A \otimes_A \cdots \otimes_A A/t_k A \\ A'/\mathcal{I}' &\cong A'/\mathcal{E}(f)(t_1)A' \otimes_{A'} A'/\mathcal{E}(f)(t_2)A' \otimes_{A'} \cdots \otimes_{A'} A'/\mathcal{E}(f)(t_k)A' \end{aligned} \quad (5.18)$$

For any $t_i \in \mathcal{E}(A)$, we know that

$$A/t_i A := \text{colim}(A \xleftarrow{t_i} A \rightarrow 0) \quad (5.19)$$

Further, we have:

$$A/t_i A \otimes_A A' = \text{colim}(A' \xleftarrow{\mathcal{E}(f)(t_i)} A' \rightarrow 0) \cong A'/\mathcal{E}(f)(t_i)A' \quad (5.20)$$

Finally, for any A -modules M and N , it is clear that

$$(M \otimes_A A') \otimes_{A'} (A' \otimes_A N) \cong M \otimes_A A' \otimes_A N \cong (M \otimes_A N) \otimes_A A' \quad (5.21)$$

Combining (5.18), (5.20) and using (5.21), it follows that $A/\mathcal{I} \otimes_A A' \cong A'/\mathcal{I}'$. \square

From Proposition 5.5 it follows that given a quasi-coherent sheaf of ideals \mathcal{I} on a Noetherian and semi-separated scheme X , the functor

$$\mathcal{O}_X/\mathcal{I} : \text{ZarAff}(X)^{op} \rightarrow \text{Comm}(\mathbf{C}) \quad \mathcal{O}_X/\mathcal{I}(U) := \mathcal{O}_X(U)/\mathcal{I}(U) \quad (5.22)$$

defines a quasi-coherent sheaf of algebras on X in the sense of Definition 5.3. We denote by $Y_{\mathcal{I}} \rightarrow X$ the affine morphism corresponding to the quasi-coherent sheaf of algebras $\mathcal{O}_X/\mathcal{I}$ as described in Proposition 5.4. We will refer to $Y_{\mathcal{I}}$ as a closed subscheme of X .

Proposition 5.6. *Let X be a Noetherian semi-separated scheme and let $Y_{\mathcal{J}}$ be a closed subscheme of X corresponding to a quasi-coherent sheaf of ideals \mathcal{J} on X . Then, $Y_{\mathcal{J}}$ is Noetherian.*

Proof. We consider a Zariski affine covering $\{U_i = \text{Spec}(A_i) \rightarrow X\}_{i \in I}$ of X . Since X is Noetherian, each A_i is Noetherian. Then, as mentioned in the proof of Proposition 5.4, the following is a pullback square:

$$\begin{array}{ccc} \text{Spec}(A_i/\mathcal{J}(U_i)) & \longrightarrow & U_i = \text{Spec}(A_i) \\ \downarrow & & \downarrow \\ Y_{\mathcal{J}} & \longrightarrow & X \end{array} \quad (5.23)$$

From Proposition 3.11, we know that each $A_i/\mathcal{J}(U_i)$ is Noetherian. Now, given any Zariski open immersion $W = \text{Spec}(B) \rightarrow Y_{\mathcal{J}}$, we consider an affine covering $\{W_{ij} = \text{Spec}(B_{ij}) \rightarrow W \times_{Y_{\mathcal{J}}} \text{Spec}(A_i/\mathcal{J}(U_i))\}_{j \in J_i, i \in I}$ of each $W \times_{Y_{\mathcal{J}}} \text{Spec}(A_i/\mathcal{J}(U_i))$, $i \in I$. Then, each $W_{ij} = \text{Spec}(B_{ij})$ admits a Zariski open immersion

$$\text{Spec}(B_{ij}) = W_{ij} \rightarrow W \times_{Y_{\mathcal{J}}} \text{Spec}(A_i/\mathcal{J}(U_i)) \rightarrow \text{Spec}(A_i/\mathcal{J}(U_i)) \quad (5.24)$$

From Proposition 3.3, it follows that each monoid B_{ij} , $j \in J_i$, $i \in I$ is Noetherian. Since the collection $\{W_{ij} = \text{Spec}(B_{ij}) \rightarrow \text{Spec}(B) = W\}_{j \in J_i, i \in I}$ is a Zariski covering, it follows from Proposition 3.5 that B is Noetherian. □

Let X be a Noetherian, integral, semi-separated scheme and let $Y_{\mathcal{J}}$ be an integral closed subscheme corresponding to a quasi-coherent sheaf of ideals \mathcal{J} on X . We will now associate to $Y_{\mathcal{J}}$ a local ring $\mathcal{O}_{Y_{\mathcal{J}}}$ that is analogous to the local ring at the generic point of an integral closed subscheme in usual algebraic geometry.

For this, we consider the collection of all pairs (U, t_U) with U an object of $\text{ZarAff}(X)$ such that $U \times_X Y_{\mathcal{J}}$ is non trivial and $t_U \in \mathcal{E}(\mathcal{O}_X(U))$. We consider two such pairs (U, t_U) and (V, t_V) . Since $U \times_X Y_{\mathcal{J}} \rightarrow Y_{\mathcal{J}}$ and $V \times_X Y_{\mathcal{J}} \rightarrow Y_{\mathcal{J}}$ are Zariski open immersions with $U \times_X Y_{\mathcal{J}}$ and $V \times_X Y_{\mathcal{J}}$ non trivial and $Y_{\mathcal{J}}$ is irreducible, it follows that

$$(U \times_X V) \times_X Y_{\mathcal{J}} = (U \times_X Y_{\mathcal{J}}) \times_{Y_{\mathcal{J}}} (V \times_X Y_{\mathcal{J}}) \quad (5.25)$$

is non trivial. Suppose that there exists a Zariski affine covering $\{W_i \rightarrow U \times_X V\}_{i \in I}$ and some $i_0 \in I$ such that $W_{i_0} \times_X Y_{\mathcal{J}}$ is non trivial and the elements in $\mathcal{E}(\mathcal{O}_X(W_{i_0}))$ corresponding to $t_U \in \mathcal{E}(\mathcal{O}_X(U))$, $t_V \in \mathcal{E}(\mathcal{O}_X(V))$ are equal. Then, we will say that

$$(U, t_U) \sim (V, t_V) \quad (5.26)$$

We note that since $W_i \times_X Y_{\mathcal{J}}$ forms a Zariski covering of $(U \times_X V) \times_X Y_{\mathcal{J}}$, there exists $\phi \neq I' \subseteq I$ such that $W_i \times_X Y_{\mathcal{J}}$ is non trivial for all $i \in I'$. Then, \sim is an equivalence relation and the collection of equivalence classes forms a ring, which we denote by $\mathcal{O}_{Y_{\mathcal{J}}}$.

Proposition 5.7. *Let X be Noetherian, integral, semi-separated scheme and let $Y_{\mathcal{J}}$ be an integral closed subscheme corresponding to a quasi-coherent sheaf of ideals \mathcal{J} on X . Then, $\mathcal{O}_{Y_{\mathcal{J}}}$ is a local ring.*

Proof. We let $\mathfrak{m} \subseteq \mathcal{O}_{Y_{\mathcal{J}}}$ denote the ideal consisting of all classes in $\mathcal{O}_{Y_{\mathcal{J}}}$ induced by pairs (U, t_U) such that $t_U \in \mathcal{J}(U)$. Then, $1 \notin \mathfrak{m}$. We now consider a pair (V, t_V) inducing a class in $\mathcal{O}_{Y_{\mathcal{J}}} \setminus \mathfrak{m}$. From Proposition 2.5, we know that $V' = \text{Spec}(\mathcal{O}_X(V)_{t_V}) \rightarrow V = \text{Spec}(\mathcal{O}_X(V))$ is a Zariski open immersion and it is clear that t_V is a unit in $\mathcal{E}(\mathcal{O}_X(V)_{t_V})$. We now consider the fibre diagrams

$$\begin{array}{ccccc} \text{Spec}((\mathcal{O}_X(V)/\mathcal{J}(V))_{t_V}) & \longrightarrow & \text{Spec}(\mathcal{O}_X(V)/\mathcal{J}(V)) & \longrightarrow & Y_{\mathcal{J}} \\ \downarrow & & \downarrow & & \downarrow \\ V' = \text{Spec}(\mathcal{O}_X(V)_{t_V}) & \longrightarrow & V = \text{Spec}(\mathcal{O}_X(V)) & \longrightarrow & X \end{array} \quad (5.27)$$

Since $Y_{\mathcal{J}}$ is integral, $\mathcal{E}(\mathcal{O}_X(V)/\mathcal{J}(V)) = \mathcal{E}(\mathcal{O}_X(V))/\mathcal{J}(V)$ is an integral domain. Since (V, t_V) induces a class in $\mathcal{O}_{Y_{\mathcal{J}}} \setminus \mathfrak{m}$, we know that $t_V \notin \mathcal{J}(V)$ and hence t_V induces a non-zero class in the integral domain $\mathcal{E}(\mathcal{O}_X(V))/\mathcal{J}(V)$. Hence, $\mathcal{E}((\mathcal{O}_X(V)/\mathcal{J}(V))_{t_V}) = (\mathcal{E}(\mathcal{O}_X(V))/\mathcal{J}(V))_{t_V} \neq 0$. Therefore, $V' \times_X Y_{\mathcal{J}} = \text{Spec}((\mathcal{O}_X(U)/\mathcal{J}(U))_{t_V})$ is non trivial.

Further, it is clear that $(V', t_V) \sim (V, t_V)$. Since t_V must be a unit in $\mathcal{E}((\mathcal{O}_X(V)/\mathcal{J}(V))_{t_V}) = (\mathcal{E}(\mathcal{O}_X(V))/\mathcal{J}(V))_{t_V}$, the pair $(V', t_V) \sim (V, t_V)$ induces a class in $\mathcal{O}_{Y_{\mathcal{J}}}$ that is a unit. Hence, any element of $\mathcal{O}_{Y_{\mathcal{J}}} \setminus \mathfrak{m}$ is a unit. It follows that $\mathcal{O}_{Y_{\mathcal{J}}}$ is a local ring with maximal ideal \mathfrak{m} . \square

6 Examples

In this section, we will present examples of categories $(\mathbf{C}, \otimes, 1)$ over which we can study algebraic geometry using the theory above. When $\mathbf{C} = R\text{-Mod}$, the category of modules over a commutative ring R , it is clear that our theory corresponds to the usual algebraic geometry of schemes over $\text{Spec}(R)$. We will now show how to construct other examples of such categories.

Let X be a topological space and let \mathcal{A} be a presheaf of commutative rings on X . We will say that a presheaf \mathcal{M} of abelian groups on X is a presheaf of \mathcal{A} -modules if it satisfies the following two conditions:

- (1) For any open $U \subseteq X$, $\mathcal{M}(U)$ is an $\mathcal{A}(U)$ module.
- (2) For open subspaces $V \subseteq U \subseteq X$, the induced morphism $\mathcal{M}(U) \rightarrow \mathcal{M}(V)$ is a morphism of $\mathcal{A}(U)$ -modules, where the $\mathcal{A}(V)$ -module $\mathcal{M}(V)$ is treated as an $\mathcal{A}(U)$ module by restriction of scalars.

Then, it is clear that the category $\text{Premod}(\mathcal{A})$ of presheaves of \mathcal{A} -modules is an abelian symmetric monoidal category. In order to show that our theory can be applied to the category $\mathbf{C} = \text{Premod}(\mathcal{A})$, we need to check that it satisfies the conditions (C1) and (C2) in Section 2.

We start by checking condition (C2). It is clear that a commutative monoid object \mathcal{B} in $\text{Premod}(\mathcal{A})$ is a presheaf of commutative \mathcal{A} -algebras; in particular, \mathcal{B} is also a presheaf of commutative rings on X . Then, the category $\mathcal{B}\text{-Mod}$ of \mathcal{B} -modules in the symmetric monoidal category $\text{Premod}(\mathcal{A})$ is identical to the category $\text{Premod}(\mathcal{B})$ of presheaves of \mathcal{B} -modules on X . From [14, Corollary 2.15], it follows that $\text{Premod}(\mathcal{B})$ is a locally finitely presented Grothendieck abelian category and hence any object in $\mathcal{B}\text{-Mod} = \text{Premod}(\mathcal{B})$ can be written as a directed colimit of finitely presented objects.

It remains to prove (C1). It is clear that the presheaf \mathcal{A} is the “unit object” for the symmetric monoidal structure on $\text{Premod}(\mathcal{A})$. Moreover, for any object $\mathcal{M} \in \mathbf{C} = \text{Premod}(\mathcal{A})$, we have

$$\text{Hom}_{\text{Premod}(\mathcal{A})}(\mathcal{A}, \mathcal{M}) \cong \mathcal{M}(X) \quad (6.1)$$

Now, let $\{\mathcal{N}_i\}_{i \in I}$ be an inductive system of objects in $\text{Premod}(\mathcal{A})$ and let $\mathcal{N} := \text{colim}_{i \in I} \mathcal{N}_i$. Then, by definition,

$$\mathcal{N}(X) = \text{colim}_{i \in I} \mathcal{N}_i(X) \quad (6.2)$$

From (6.1) and (6.2), it follows that

$$\text{colim}_{i \in I} \text{Hom}_{\text{Premod}(\mathcal{A})}(\mathcal{A}, \mathcal{N}_i) \cong \text{Hom}_{\text{Premod}(\mathcal{A})}(\mathcal{A}, \text{colim}_{i \in I} \mathcal{N}_i) \quad (6.3)$$

In particular, I could be a filtered inductive system or a finite system. Hence, $\mathbf{C} = \text{Premod}(\mathcal{A})$ satisfies condition (C1) as well.

Hence, given a topological space X and a presheaf \mathcal{A} of commutative rings on X , the above theory enables us to do algebraic geometry in the category $\text{Premod}(\mathcal{A})$ of presheaves of \mathcal{A} -modules on X . We end by mentioning several natural examples of such situations.

(1) Let X be a topological space and let R be a commutative ring. We can take \mathcal{A} to be the constant presheaf of rings R on X . Then, the category $\text{Premod}(\mathcal{A})$ is the category of presheaves of R -modules on X .

(2) Let X be a scheme. We can choose \mathcal{A} to be the structure sheaf \mathcal{O}_X of X . Then, the category $\text{Premod}(\mathcal{A})$ is the category of presheaves of \mathcal{O}_X -modules on X .

(3) Let X be a topological space. We can define a presheaf $\mathcal{A}_{\mathbb{R}}$ (resp. a presheaf $\mathcal{A}_{\mathbb{C}}$) of rings on X by setting $\mathcal{A}_{\mathbb{R}}(U)$ (resp. $\mathcal{A}_{\mathbb{C}}(U)$) to be the ring of continuous real valued (resp. complex valued) functions on U , for any open set $U \subseteq X$.

(4) Let X be a smooth (resp. complex) manifold. We can consider the presheaf $\mathcal{A}_{\mathbb{R}}^{\infty}$ (resp. $\mathcal{A}_{\mathbb{C}}^{\infty}$) of rings by setting $\mathcal{A}_{\mathbb{R}}^{\infty}(U)$ (resp. $\mathcal{A}_{\mathbb{C}}^{\infty}(U)$) to be the ring of infinitely differentiable real valued (resp. holomorphic complex valued) functions on U , for any open set $U \subseteq X$.

(5) Let X be a scheme of finite type over \mathbb{C} . Then, using the GAGA principle, any Zariski open U in X corresponds to an analytic space U^{an} . Further, since X is a scheme of finite type, this association is functorial. Hence, we can consider the presheaf \mathcal{A}^{an} of rings defined by setting $\mathcal{A}^{an}(U)$ to be the ring of continuous complex valued functions on U^{an} for any Zariski open U in X .

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